QUEUES WITH BALKING AND THEIR APPLICATION TO AN INVENTORY PROBLEM

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TO THE MEMORY OF

MY MOTHER

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CHAPTER 1

INTRODUCTION

In this dissertation, we consider an alternative to the (s,S) ordering policy associated with inventory systems.

The (s,S) ordering policy is specified as follows. There exists a store of finite capacity S that holds material (discrete or continuous) for future use in some process. In the most general context, demand for the material in storage during an interval of time is assumed to be a time-dependent stochastic process. Ordering of replacement stock to maintain the level of inventory in the store is done in one of two ways. Either orders for an amount S-s of replacement stock are made at the times when the stock level reaches s, $s \leq S$, or the level of stock in the inventory is examined at regular points in time and orders for replacement stock equal to the stock deficit are only made at those regular times for which the stock has fallen below the level s. In both cases, the time it takes the replacement stock to arrive (i.e., the delivery time) is assumed to be zero. A generalization of the (s,S) ordering policy allows a time lag T for arrival of the replacement stock.

For a certain class of cost functions associated with maintaining the level of stock in an inventory, it can be shown that the (s,S) ordering policy is the optimal policy to utilize. A summary of some results

for the (s,S) ordering policy and conditions under which (s,S) is (or is not) the optimal ordering policy is given in a paper by Gani (1957).

Another generalization of the (s,S) ordering policy is the following. The capacity of the inventory is S and demand for the stored material is once again a time-dependent stochastic process. However, orders for an amount ν ($\nu \leq S$) of replacement stock are now made at the times when the stock level drops to the values $S-\nu, S-2\nu, S-3\nu, \ldots$. The delivery time for any order is assumed to be a constant value T ($T \geq 0$). Under the assumptions that the store holds discrete items and the demand for these items obeys a Poisson probability law, the long run probability law representing the level of stock in the store is given in Gani (1957) and Prabhu (1965b).

In many cases, however, a constant delivery time does not adequately express reality. Furthermore, the negative stock level that can arise when T (the delivery time) is greater than zero may reflect the loss of considerable time and money in terms of idle manpower and equipment. To circumvent these difficulties, an alternative ordering policy is defined and its properties examined in this paper.

Envision a subwarehouse, maintaining an inventory of finite capacity S, that holds material (discrete or continuous) for future use in some process. In the most general context, we assume the demand for the stored material is a time-dependent stochastic process. In order to maintain a stock on hand, orders for an amount ν ($\nu \leq S$) of replacement stock are placed with a warehouse at the times when the stock level drops to $S-\nu, S-2\nu, \ldots, S-\nu[S/\nu]$ ($[x]$ the integral part of x). The time it takes

the warehouse to process an order placed when the stock level falls to $S-\nu, S-2\nu, \ldots,$ or $S-\nu[S/\nu]+\nu$ is called a regular service time. All regular service times are assumed to be mutually independent random variables with a common distribution and to be independent of the demand process. An order placed when the stock level falls to $S-\nu[S/\nu]$ is called an emergency order. The time to process an emergency order is assumed to be instantaneous, or at least effectively zero.

Hence, regular orders for an amount ν of replacement stock are made if the stock level is at least ν at the instant an order is placed, while an emergency order is made if the stock level is less than ν at the instant the order is placed. Utilizing this reordering technique, the inventory maintains a positive stock level at all times.

Although a somewhat larger cost would quite naturally be incurred with emergency orders than with regular ones, it is assumed we are willing to pay the price of instantaneous delivery in order to avoid the disaster of running completely out of stock in the inventory.

The cost of maintaining the inventory level will clearly depend on ν, the size of a replacement order, and there should exist an optimal value of ν, defined to be that value of ν for which this cost is a minimum. In Chapter 5, we define a ν-dependent cost function for which we seek the optimal ν.

It will be shown later that the inventory problem is closely related to a problem in queueing theory--queueing systems with balking at queues of a fixed length. We shall now discuss the salient features of such a queueing system.

Utilizing a notation proposed by Kendall (1953), by "the queue A/B/s with balking at queues of length K-1" we mean a queueing system specified as follows. The queue length at any instant will refer to the number of people in the system who are being served or waiting to be served at that instant. Successive customers are assumed to arrive in the system in such a way that their inter-arrival times are mutually independent with distribution function $A(\cdot)$. A customer joins the queue if, at the instant he arrives, there are less than K-1 persons already in the queue. If there are K-1 persons in the queue when the customer arrives (so that he is the K-th person in the system), one of three equivalent things happens to him: (1) The customer balks, i.e., he leaves without waiting to be served; (2) The system rejects the customer; or (3) The customer receives instantaneous service. There are s servers available to wait on customers with the first free server attending the customer at the top of the queue. The length of time from when a server starts to serve a customer until the completion of such service is called the service time. All service times are assumed to be mutually independent with distribution function $B(\cdot)$. Finally, the service times and inter-arrival times are assumed to be mutually independent.

Because the statistician is more familiar with the terminology of queues rather than inventories, the work has been carried out in terms of queueing theory. The times between successive orders and the service times for the inventory problem with emergency orders are shown to correspond to the inter-arrival times and service times, respectively, in queueing systems with balking at queues of a fixed length. The mechanics

of inventories have led us to give prime attention to the queues GI/M/1, GI/M/∞, and GI/D/1, all with balking at queues of length K-1, where GI (or G) refers to a general distribution function, M refers to a negative exponential distribution function, and D refers to a distribution whose mass is concentrated at a single point.

An inventory or storage area is normally established with the assumption that it will be in operation for a long period of time. In choosing a reordering policy, therefore, long run distributions become important. Fortunately, this means that long run properties of queues with balking are adequate for the solution of our inventory problem.

In Chapter 2, the queue GI/M/1 with balking at queues of length K-1 is discussed. In particular, we utilize the concept of an imbedded Markov chain to derive properties of the queue length and the time between successive balks. In Chapter 3, the same is done for the queue GI/M/∞ with balking at queues of length K-1.

In Chapter 4, the queue GI/D/1 with balking at queues of length K-1 is discussed. Here, the concept of the waiting time in the system is introduced. We again utilize an imbedded Markov process to obtain properties of the waiting time and the time between successive balks.

In Chapter 5, we consider the inventory problem when the stored material is discrete. Here, we forge the link between queues with balking and inventories subject to instantaneous emergency orders, and give solutions for the cost function associated with the inventory problem based on results from Chapters 2 through 4, along with some examples.

In Chapter 6, the continuous inventory problem and its relation to results of previous chapters is discussed. Also included are some properties of a continuous, non-negative stochastic process with an infinitely divisible distribution.

CHAPTER 2

THE QUEUE GI/M/1 WITH BALKING AT QUEUES OF LENGTH K-1

2.1 The Basic System

Consider a queueing system in which customers arrive in the system at times, $\ldots,\sigma_{-2},\sigma_{-1},\sigma_0,\sigma_1,\sigma_2,\ldots$, such that the inter-arrival times

$$u_j = \sigma_j - \sigma_{j-1}, \quad j \geq 1, \tag{2.1.1}$$

are mutually independent. The distribution function of u_j will be denoted by

$$\Pr\{u_j \leq u\} = F(u), \quad u \geq 0, \quad j \geq 1. \tag{2.1.2}$$

One server is available to handle the needs of the customers. This server dispenses his service on a strict "first come, first served" basis. The successive service times of customers who join the queue are denoted by $w_1,w_2,w_3,\ldots$, and are assumed to be mutually independent random variables that are independent of the arrival times. The distribution function of w_j is assumed to be

$$\Pr\{w_j \leq w\} = 1 - e^{-\lambda w}, \quad w \geq 0, \quad j \geq 1. \tag{2.1.3}$$

Let $\{Q(t); -\infty<t<+\infty\}$ be the stochastic process such that $Q(t)$ represents the number of customers in the system at time t and $\{Q^*(t); -\infty < t < +\infty\}$ the stochastic process such that $Q^*(t)$ represents

the queue length at time t. Recall that the queue length, $Q^*(t)$, is the number of persons being served or waiting to be served at time t.

A customer arriving in the system at time σ enters the queue if and only if $Q^*(\sigma-0) \leq K-2$, that is, if and only if the number of people in the queue immediately prior to his arrival is K-2 or less. In this case we have

$$Q^*(\sigma) = Q(\sigma) = Q^*(\sigma-0) + 1 = Q(\sigma-0) + 1.$$

If, on the other hand, our customer is faced with a queue length of K-1 (so that he becomes the K-th person in the system), he balks and immediately leaves the system. We now have that $Q(\sigma) = K$ implies

$$Q(\sigma-0) = Q(\sigma+0) = K-1$$

and

$$Q^*(\sigma-0) = Q^*(\sigma+0) = Q^*(\sigma) = K-1.$$

It is clear that $Q(t)$ and $Q^*(t)$ are identical in value except at the points on the time axis for which $Q(\sigma) = K$.

We shall work with the stochastic process $\{Q(t); -\infty < t < +\infty\}$ and shall be concerned with its behavior beyond the time point σ_o, which we assume is known. Hence, without loss of generality, σ_o could be taken as zero.

Define

$$\begin{aligned} N_1 &= \inf \{k > 0 \,|\, Q(\sigma_k) = K\}, \\ N_n &= \inf \{k > N_{n-1} \,|\, Q(\sigma_k) = K\}, \quad n \geq 2, \end{aligned} \tag{2.1.4}$$

(so that N_k is the number of customers who arrive up to and including the n-th customer to balk), and

$$M_1 = N_1,$$
$$M_j = N_j - N_{j-1}, \quad j \geq 2, \tag{2.1.5}$$

(so that M_j $(j \geq 2)$ is the number of arrivals between the (j-1)-st and j-th balks plus the j-th person to balk).

Define

$$V_1 = \sigma_{N_1} - \sigma_0$$
$$V_j = \sigma_{N_j} - \sigma_{N_{j-1}}, \quad j \geq 2. \tag{2.1.6}$$

Then V_1 is the time until the first balk and V_j $(j \geq 2)$ is the time between the (j-1)-st and j-th balks.

Of primary importance, for us, is the value of $\mathcal{E}(V_j)$, $j \geq 2$. This quantity is established in Section 2.3. While the theorems of Sections 2.2 and 2.4 are proved with the thought of building toward a solution to $\mathcal{E}(V_j)$, these theorems have a theoretical and practical importance that goes beyond our narrow objective.

2.2 An Imbedded Markov Chain

Before we define the imbedded Markov chain, we prove two lemmas. The first lemma simply restates a well-known result about negative exponentially distributed random variables, while the second lemma establishes the non-Markovian character of Q(t) when the inter-arrival times have an unspecified distribution.

Lemma 2.2.1

Let X be a continuous, non-negative random variable. Then

$$\Pr\{X > x + y | X > x\} = \Pr\{X > y\}, \quad x,y \geq 0, \tag{2.2.1}$$

if and only if

$$\Pr\{X > x\} = e^{-\mu x}, \quad \mu > 0. \tag{2.2.2}$$

Proof of Lemma 2.2.1

If (2.2.1) holds, we have

$$\Pr\{X > x + y\} = \Pr\{X > x\} \Pr\{X > y\}$$

and therefore (2.2.2) is true. See, for example, Parzen (1962, p. 121).

If (2.2.2) holds, we have

$$\begin{aligned}\Pr\{X > x + y | X > x\} &= \Pr\{X > x + y, X > x\}/\Pr\{X > x\} \\ &= \Pr\{X > x + y\}/\Pr\{X > x\} \\ &= e^{-\mu(x+y)}/e^{-\mu x} \\ &= e^{-\mu y} = \Pr\{X > y\},\end{aligned}$$

thus completing the proof.

Lemma 2.2.2

The stochastic process $\{Q(t); -\infty < t < +\infty\}$ is not, in general, a Markov process.

Proof of Lemma 2.2.2

Without loss of generality, let $\sigma_o = 0$ and $Q(0) = 0$.

Let $Y(t) = \max_{s \le \tau \le t} Q(\tau)$ and $u = t-s$. We have

$$\begin{aligned}\Pr\{Y(t) = 0 | Q(\tau) = 0,\ 0 \le \tau \le s\} &= \Pr\{Y(t) = 0 | u_1 > s\} \\ &= \Pr\{u_1 > t | u_1 > s\} \\ &= \Pr\{u_1 > u + s | u_1 > s\}. \end{aligned} \tag{2.2.3}$$

Let $0 < \tau_o < \tau_1 < s$, and define the events A and B by

$$A = \{Q(\tau) = 0;\ 0 \le \tau < \tau_o,\ \tau_1 \le \tau \le s\}$$

and

$$B = \{Q(\tau) = 1;\ \tau_o \le \tau < \tau_1\}\ .$$

We have

$$\begin{aligned}
&\Pr\{Y(t) = 0 \mid A,B\} \\
&\quad = \Pr\{Y(t) = 0 \mid u_1 = \tau_o,\ u_1 + w_1 = \tau_1,\ u_1 + u_2 > s\} \\
&\quad = \Pr\{u_1 + u_2 > t \mid u_1 = \tau_o,\ u_1 + w_1 = \tau_1,\ u_1 + u_2 > s\} \\
&\quad = \Pr\{u_2 > t - \tau_o \mid u_2 > s - \tau_o\} \\
&\quad = \Pr\{u_1 > t - \tau_o \mid u_1 > s - \tau_o\} \\
&\quad = \Pr\{u_1 > u + (s - \tau_o) \mid u_1 > s - \tau_o\}. \qquad (2.2.4)
\end{aligned}$$

But (2.2.3) and (2.2.4) are not necessarily equivalent.

Yet, if

$$\Pr\{Q(t) = k \mid Q(\tau),\ 0 \le \tau \le s\} = \Pr\{Q(t) = k \mid Q(s)\}$$

for all $t \ge s$, then (2.2.3) and (2.2.4) would have to be identical. Therefore, the proof is complete.

It should be noted here, that, if the u_j have a negative exponential distribution, then Q(t) is a Markov process.

Although $\{Q(t);\ -\infty < t < +\infty\}$ is a non-Markovian process (in general), there exists an imbedded Markov chain defined by

$$Q_n = Q(\sigma_n),\quad n = 0,1,2,\ldots, \qquad (2.2.5)$$

regardless of the distribution of the u_j.

Figures 2.1 and 2.2 give the correspondence between Q(t) and Q_n. Q_n clearly represents the number of persons in the system at the time the n-th customer enters the system. Valuable information about Q(t)

can be obtained from a knowledge of Q_n as will be shown in the following theorems.

Before proceeding further, note that if $K = 1$, then $\Pr\{Q_n = 1\} = 1$, $n = 0,1,2,\ldots$. For the future we shall therefore hold $K \geq 2$.

Theorem 2.2.1

The stochastic process $\{Q_n;\ n = 0,1,2,\ldots\}$ defined by (2.2.5) has the following properties:

(a) Q_n is a Markov chain;

(b) Q_n is time-homogeneous;

(c) The class $\{1,2,\ldots,K\}$ of states on which Q_n is defined is an aperiodic, positive persistent communicating class; and

(d) The one-step transition probability matrix P is given by

↗	K	K-1	K-2	...	3	2	1
K	α_0	α_1	α_2	...	α_{K-3}	α_{K-2}	k_{K-2}
K-1	α_0	α_1	α_2	...	α_{K-3}	α_{K-2}	k_{K-2}
K-2	0	α_0	α_1	...	α_{K-4}	α_{K-3}	k_{K-3}
.	.	.	.		.	.	.
.	.	.	.		.	.	.
.	.	.	.		.	.	.
3	0	0	0	...	α_1	α_2	k_2
2	0	0	0	...	α_0	α_1	k_1
1	0	0	0	...	0	α_0	k_0

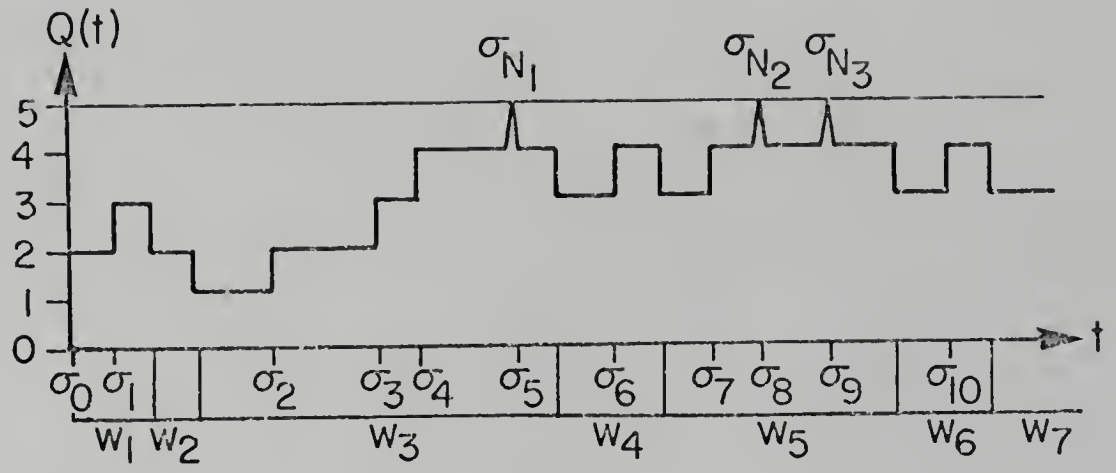

Figure 2.1. A Typical Path of Q(t) for GI/M/1 with Balking at Queues of Length 4.

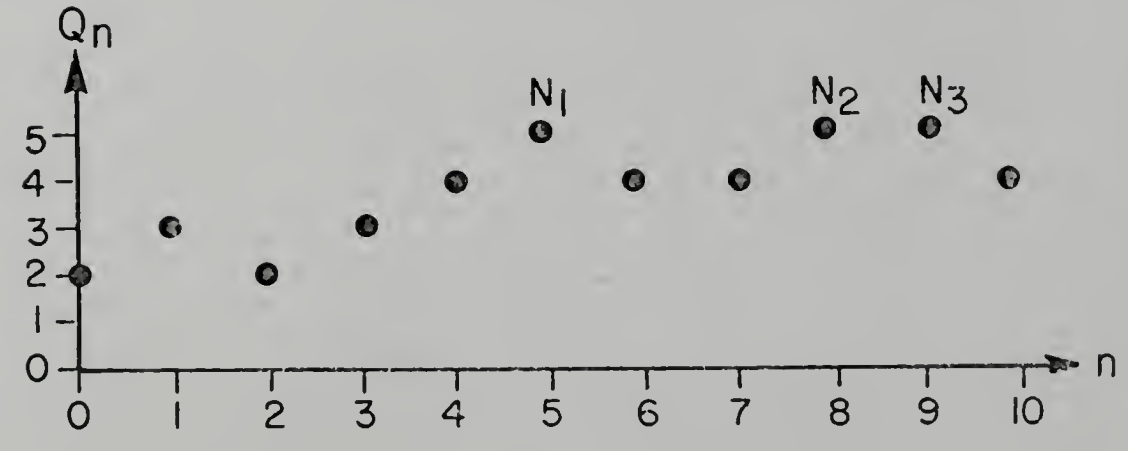

Figure 2.2. Path of Q_n Corresponding to Q(t) in Figure 2.1.

where

$$\alpha_j = \int_o^\infty \frac{(\lambda t)^j}{j!} e^{-\lambda t} \, dF(t), \quad j = 0,1,2,\ldots, \tag{2.2.6}$$

and

$$k_j = 1 - \alpha_0 - \alpha_1 - \ldots - \alpha_j, \quad j = 0,1,2,\ldots. \tag{2.2.7}$$

Proof of Theorem 2.2.1, Part (a)

Let $U(\cdot)$ be the unit step function at zero. Let X_{n+1} be the number of customers who complete their service in the interval $(\sigma_n, \sigma_{n+1}]$. Then

$$Q_{n+1} = Q_n + 1 - X_{n+1}, \qquad \text{if } Q_n < K,$$

$$Q_{n+1} = Q_n + 1 - X_{n+1} - 1, \qquad \text{if } Q_n = K,$$

so that

$$Q_{n+1} = Q_n - X_{n+1} - U(Q_n - K) + 1, \qquad Q_n \leq K.$$

Since the distribution on service times is negative exponential, the probability law on X_{n+1} conditional on the history $Q_0, Q_1, \ldots, Q_n$ is a function only of Q_n. Hence, we see easily that the probability law on Q_{n+1} conditional on the history $Q_0, Q_1, \ldots, Q_n$ can be a function only of Q_n, which establishes the Markov property of Q_n.

Proof of Theorem 2.2.1, Parts (b) and (d)

Since we are looking at $Q(t)$ at successive arrivals, we have $1 \leq Q_{n+1} \leq Q_n + 1$. Also, of course, by the balking aspect of the problem $Q_n \leq K$. Hence.

$$\Pr\{Q_{n+1} = k \mid Q_n = j\} = 0, \quad \text{for} \quad k > j + 1. \tag{2.2.8}$$

Further,

$$\Pr\{Q_{n+1} = k \mid Q_n = K\} = \Pr\{K - X_{n+1} - 1 + 1 = k\}$$
$$= \Pr\{(k-1) - X_{n+1} + 1 = k\}$$
$$= \Pr\{Q_{n+1} = k \mid Q_n = K - 1\}. \qquad (2.2.9)$$

For $j = 1,2,\ldots,K-1$ and $k = 2,\ldots,j+1$, let $N(t)$ be a Poisson process with inter-arrival times $w_1, w_2, w_3, \ldots$, then

$$\Pr\{Q_{n+1} = k \mid Q_n = j\} = \Pr\{j - X_{n+1} + 1 = k \mid Q_n = j\}$$
$$= \Pr\{X_{n+1} = j - k + 1 \mid Q_n = j\}$$
$$= \mathcal{E}_{u_{n+1}}(\Pr\{w_1 + \ldots + w_{j-k+1} \le u_{n+1},\ w_1 + \ldots + w_{j-k+2} > u_{n+1}\})$$
$$= \mathcal{E}_{u_{n+1}}(\Pr\{N(u_{n+1}) = j - k + 1\})$$
$$= \int_0^\infty e^{-\lambda t} \frac{(\lambda t)^{j-k+1}}{(j-k+1)!}\, dF(t)$$
$$= \alpha_{j-k+1}. \qquad (2.2.10)$$

Let $j = 1,2,3,\ldots,K-1$ and $k = 1$, then

$$\Pr\{Q_{n+1} = 1 \mid Q_n = j\} = \Pr\{j - X_{n+1} + 1 = 1 \mid Q_n = j\}$$
$$= \Pr\{X_{n+1} = j \mid Q_n = j\}$$
$$= \mathcal{E}_{u_{n+1}}(\Pr\{w_1 + \ldots + w_j \le u_{n+1}\})$$
$$= \int_0^\infty \left(\int_0^t x^{j-1} \lambda^j e^{-\lambda x}/(j-1)!\, dx\right) dF(t)$$
$$= \int_0^\infty \left(1 - \sum_{i=0}^{j-1} e^{-\lambda t} (\lambda t)^i / i!\right) dF(t)$$
$$= 1 - \alpha_0 - \ldots - \alpha_{j-1} = k_{j-1}. \qquad (2.2.11)$$

Equations (2.2.8) through (2.2.11) are independent of n and therefore Q_n is time-homogeneous. Application of these equations for $j,k = 1,2,\ldots K$ gives us the matrix P.

Proof of Theorem 2.2.1, Part (c)

An examination of the one-step transition matrix in the statement of the theorem shows that each state communicates with all others.

Since, for example,

$$\Pr\{Q_{n+1} = K | Q_n = K\} > 0,$$

state K is aperiodic. We therefore have a finite irreducible communicating class of aperiodic states so that each state is necessarily positive persistent. The proof of Theorem 2.2.1 is now complete.

We shall prove a lemma that applies to an arbitrary Markov chain with one-step transition matrix P^* given by

↗	K	K-1	K-2	...	3	2	1
K	$p_{K-1,1}$	$p_{K-1,2}$	$p_{K-1,3}$	⋯	$p_{K-1,K-2}$	$p_{K-1,K-1}$	$p_{K-1,K}$
K-1	$p_{K-1,1}$	$p_{K-1,2}$	$p_{K-1,3}$	⋯	$p_{K-1,K-2}$	$p_{K-1,K-1}$	$p_{K-1,K}$
K-2	0	$p_{K-2,1}$	$p_{K-2,2}$	⋯	$p_{K-2,K-3}$	$p_{K-2,K-2}$	$p_{K-2,K-1}$
.	.	.	.		.	.	.
.	.	.	.		.	.	.
.	.	.	.		.	.	.
3	0	0	0	...	$p_{3,2}$	$p_{3,3}$	$p_{3,4}$
2	0	0	0	...	$p_{2,1}$	$p_{2,2}$	$p_{2,3}$
1	0	0	0	...	0	$p_{1,1}$	$p_{1,2}$

(2.2.12)

in which $p_{i,j} > 0$ for $i = 1,2,\ldots,K-1$; $j = 1,2,\ldots,i+1$.

If a Markov chain has a one-step transition matrix of the form (2.2.12), then clearly the Markov chain is aperiodic and positive persistent. Hence, there exists a unique long run distribution equal to the stationary distribution. With this in mind, we now state and prove the lemma.

Lemma 2.2.3

Let P^* of (2.2.12) be the one-step transition probability matrix of a Markov chain. Then, if $\underline{\theta}' = (\theta_K,\ldots,\theta_1)$ is the unique stationary distribution for the Markov chain,

$$1/\theta_K = \xi_K - \xi_{K-1},$$

where

$$(\xi_K,\xi_{K-1},\ldots,\xi_2)B = (1,1,\ldots,1)$$

and

$$B = \begin{bmatrix} p_{K-1,1} & & & & & & \\ p_{K-1,2}^{-1} & p_{K-2,1} & & & & & \\ p_{K-1,3} & p_{K-2,2}^{-1} & p_{K-3,1} & & & & \\ \vdots & \vdots & \vdots & & & & \\ p_{K-1,K-2} & p_{K-2,K-3} & p_{K-3,K-4} & \cdots & p_{3,2}^{-1} & p_{2,1} & \\ p_{K-1,K-1} & p_{K-2,K-2} & p_{K-3,K-3} & \cdots & p_{3,3} & p_{2,2}^{-1} & p_{1,1} \end{bmatrix}.$$

Proof of Lemma 2.2.3

By the definition of a stationary distribution, $\underline{\theta}$ is the unique solution to

$$\underline{\theta}' P^{*} = \underline{\theta}' \tag{2.2.13}$$

and

$$\theta_1 + \ldots + \theta_K = 1 \ . \tag{2.2.14}$$

Writing out the equations (2.2.13) and (2.2.14) with θ_K on the left hand side, we get the following system of K linearly independent equations:

$$\begin{aligned}
(1-p_{K-1,1})\theta_K &= p_{K-1,1}\theta_{K-1} \\
-p_{K-1,2}\theta_K &= (p_{K-1,2}-1)\theta_{K-1} + p_{K-2,1}\theta_{K-2} \\
-p_{K-1,3}\theta_K &= p_{K-1,3}\theta_{K-1} + (p_{K-2,2}-1)\theta_{K-2} + p_{K-3,1}\theta_{K-3} \\
&\ \ \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
-p_{K-1,K-2}\theta_K &= p_{K-1,K-2}\theta_{K-1} + p_{K-2,K-3}\theta_{K-2} + \ldots + (p_{3,2}-1)\theta_3 + p_{2,1}\theta_2 \\
-p_{K-1,K-1}\theta_K &= p_{K-1,K-1}\theta_{K-1} + p_{K-2,K-2}\theta_{K-2} + \ldots \\
&\qquad\qquad + p_{3,3}\theta_3 + (p_{2,2}-1)\theta_2 + p_{1,1}\theta_1 \\
-\theta_K &= \theta_{K-1} + \ldots + \theta_1 - 1 .
\end{aligned} \tag{2.2.15}$$

Let A be the (K x K-1) matrix of coefficients of $\theta_{K-1},\ldots,\theta_1$ in (2.2.15), then

$$A = \begin{bmatrix} p_{K-1,1} & & & & & & \\ p_{K-1,2}^{-1} & p_{K-2,1} & & & & & \\ p_{K-1,3} & p_{K-2,2}^{-1} & p_{K-3,1} & & & & \\ \vdots & \vdots & \vdots & & & & \\ p_{K-1,K-2} & p_{K-2,K-3} & p_{K-3,K-4} & \cdots & p_{3,2}^{-1} & p_{2,1} & \\ p_{K-1,K-1} & p_{K-2,K-2} & p_{K-3,K-3} & \cdots & p_{3,3} & p_{2,2}^{-1} & p_{1,1} \\ 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix} .$$

The K-1 columns of A form a set of linearly independent vectors of dimension K. Hence, there exists a vector $\underline{\xi}' = (\xi_K, \xi_{K-1}, \ldots, \xi_1)$ such that $\underline{\xi}$ is orthogonal to each column of A. By noting the last column of A, we must have $\xi_1 \neq 0$. Therefore, without loss of generality, we may take $\xi_1 = -1$. That is, we have

$$\underline{\xi}'A = \underline{0}' \tag{2.2.16}$$

with $\underline{0}$ the null vector.

After multiplying the n-th equation of (2.2.15) by ξ_n and adding them up, we find

$$\theta_K[(1-p_{K-1,1})\xi_K - p_{K-1,2}\xi_{K-1} - p_{K-1,3}\xi_{K-2} - \cdots - p_{K-1,K-2}\xi_3 - p_{K-1,K-1}\xi_2 + 1] = 1. \tag{2.2.17}$$

But by (2.2.16), if we take the scalar product of $\underline{\xi}$ with the first column of A, we have

$$p_{K-1,1}\xi_K + (p_{K-1,2}^{-1})\xi_{K-1} + p_{K-1,3}\xi_{K-2} + \cdots + p_{K-1,K-1}\xi_2 - 1 = 0. \tag{2.2.18}$$

Substituting (2.2.18) into (2.2.17) yields

$$\theta_K(\xi_K - \xi_{K-1}) = 1.$$

Since

$$A = \begin{bmatrix} & B & \\ \cdots & \cdots & \cdots \\ 1, & 1, \ldots, & 1 \end{bmatrix}$$

and $\xi_1 = -1$, (2.2.16) implies that

$$(\xi_K, \xi_{K-1}, \ldots, \xi_2)B = (1, 1, \ldots, 1),$$

thus completing the proof.

Corollary 2.2.1

Let μ_{KK} be the mean recurrence time of state K for a Markov chain with one-step transition matrix P^* of (2.2.12) and state space $\{1,2,..,K\}$. Then $\mu_{22}, \mu_{33}, \mu_{44}, \ldots,$ are finite and satisfy

$$\mu_{KK} = \begin{cases} 1/p_{1,1}, & K=2, \\ (1/p_{K-1,1})\left[1 + \sum_{k=2}^{K-1} p_{K-1,K-k+2} \sum_{j=k}^{K-1} \mu_{jj}\right], & K \geq 3. \end{cases} \tag{2.2.19}$$

Proof of Corollary 2.2.1

It is well known that $\mu_{KK} = 1/\theta_K$, θ_K the stationary probability associated with state K, and μ_{KK} is finite. See, for example, Parzen (1962).

By Lemma 2.2.3, if K = 2, then $B = [p_{1,1}]$ and hence

$$\mu_{22} = 1/\theta_2 = 1/p_{1,1} = \xi_2. \tag{2.2.20}$$

If $K \geq 3$, take the scalar product of the first column of B and $(\xi_K, \ldots, \xi_2)$, then the lemma yields

$$p_{K-1,1}\xi_K = (1-p_{K-1,2})\xi_{K-1} - p_{K-1,3}\xi_{K-2} - p_{K-1,4}\xi_{K-3} - \ldots - p_{K-1,K-2}\xi_3 - p_{K-1,K-1}\xi_2 + 1. \quad (2.2.21)$$

Subtracting $p_{K-1,1}\xi_{K-1}$ from both sides of equation (2.2.21), we get

$$p_{K-1,1}(\xi_K - \xi_{K-1}) = (1 - p_{K-1,2} - p_{K-1,1})\xi_{K-1} - p_{K-1,3}\xi_{K-2} - \ldots - p_{K-1,K-1}\xi_2 + 1. \quad (2.2.22)$$

Adding and subtracting $(1 - p_{K-1,2} - p_{K-1,1})\xi_{K-2}$ on the right hand side of equation (2.2.22) we obtain

$$p_{K-1,1}(\xi_K - \xi_{K-1}) = (1 - p_{K-1,2} - p_{K-1,1})(\xi_{K-1} - \xi_{K-2}) + (1 - p_{K-1,3} - p_{K-1,2} - p_{K-1,1})\xi_{K-2} - p_{K-1,4}\xi_{K-3} - \ldots - p_{K-1,K-1}\xi_2 + 1. \quad (2.2.23)$$

Continuing in this manner to add and subtract (coefficient of $\xi_j)\cdot\xi_{j-1}$ as j goes from K-2 to 3, we get

$$\begin{aligned} p_{K-1,1}(\xi_K - \xi_{K-1}) &= (1 - p_{K-1,2} - p_{K-1,1})(\xi_{K-1} - \xi_{K-2}) \\ &+ (1 - p_{K-1,3} - p_{K-1,2} - p_{K-1,1})(\xi_{K-2} - \xi_{K-3}) \\ &+ \ldots + (1 - p_{K-1,K-2} - p_{K-1,K-3} - \cdots - p_{K-1,1})(\xi_3 - \xi_2) \\ &+ (1 - p_{K-1,K-1} - \cdots - p_{K-1,1})\xi_2 + 1. \end{aligned} \quad (2.2.24)$$

Note that $1 - \sum_{k=1}^{n} p_{K-1,k} = \sum_{k=n+1}^{K} p_{K-1,k}$.

Applying this result, equation (2.2.20), and $\mu_{jj} = \xi_j - \xi_{j-1}$ to (2.2.24), we have

$$p_{K-1,1}\mu_{KK} = 1 + \sum_{j=2}^{K-1} \mu_{jj} \sum_{k=K-j+1}^{K-1} p_{K-1,k+1}. \tag{2.2.25}$$

Setting $i = (K-1) - k+2$ in equation (2.2.25) yields

$$p_{K-1,1}\mu_{KK} = 1 + \sum_{j=2}^{K-1} \mu_{jj} \sum_{i=2}^{j} p_{K-1,K-i+2}$$

$$= 1 + \sum_{i=2}^{K-1} p_{K-1,K-i+2} \sum_{j=i}^{K-1} \mu_{jj}. \tag{2.2.26}$$

Dividing both sides of (2.2.26) by $p_{K-1,1}$ yields equation (2.2.19), thus completing the proof.

The result of Corollary 2.2.1 can be applied to the Markov chain of Theorem 2.2.1 to obtain the mean recurrence time of state K. However, the special form of the matrix P in Theorem 2.2.1 lends itself to a more elegant solution for the mean recurrence time than that given by the corollary. Before stating this solution in Theorem 2.2.2, we define

$$\varphi(\theta) = \int_0^\infty e^{-\theta u}\, dF(u), \qquad \theta > 0, \tag{2.2.27}$$

and

$$K(Z) = \sum_{j=0}^{\infty} \alpha_j Z^j, \qquad |Z| \le 1. \tag{2.2.28}$$

Also, for ease in writing, if $h(Z)$ is any function of Z, denote by $C_Z^{(n)} h(Z)$ the coefficient of Z^n in the expression $h(Z)$.

Theorem 2.2.2

The mean recurrence time μ_{KK} for the state K of the Markov process $\{Q_n;\ n = 0,1,2,\ldots\}$ of Theorem 2.2.1 is finite and satisfies

$$\mu_{KK} = C_Z^{(K-2)}\ 1/[\varphi(\lambda(1 - Z)) - Z]. \tag{2.2.29}$$

Proof of Theorem 2.2.2

By Theorem 2.2.1, μ_{KK} is finite and

$$\mu_{KK} = 1/\theta_K, \tag{2.2.30}$$

where θ_K is the stationary probability associated with state K.

The P of Theorem 2.2.1(d) is P^* of (2.2.12) with

$$P_{i,j} = \begin{cases} \alpha_{i-1}, & j \geq i, \\ k_{j-1}, & j = i - 1. \end{cases}$$

Hence, Lemma 2.2.3 applies with

$$B = \begin{bmatrix} \alpha_0 & & & & & & \\ (\alpha_1 - 1) & \alpha_0 & & & & & \\ \alpha_2 & (\alpha_1 - 1) & \alpha_0 & & & & \\ \vdots & \vdots & \vdots & & & & \\ \alpha_{K-3} & \alpha_{K-4} & \alpha_{K-5} & \cdots & (\alpha_1 - 1) & \alpha_0 & \\ \alpha_{K-2} & \alpha_{K-3} & \alpha_{K-4} & \cdots & \alpha_2 & (\alpha_1 - 1) & \alpha_0 \end{bmatrix}.$$

In Section 2.4, it will be shown that for a matrix of the special form B above, the solution of $1/\theta_K$ in Lemma 2.2.3 is

$$1/\theta_K = \xi_K - \xi_{K-1} = C_Z^{(K-2)} \; 1/(K(Z)-Z). \tag{2.2.31}$$

But

$$\left| \sum_{j=0}^{\infty} e^{-\lambda t} (\lambda Z t)^j / j! \right| \leq 1, \quad \text{if} \quad |Z| \leq 1.$$

Therefore,

$$\begin{aligned} K(Z) &= \sum_{j=0}^{\infty} Z^j \int_0^{\infty} e^{-\lambda t} (\lambda t)^j / j! \; dF(t) \\ &= \int_0^{\infty} \sum_{j=0}^{\infty} e^{-\lambda t} (\lambda Z t)^j / j! \; dF(t) \\ &= \int_0^{\infty} e^{-\lambda t(1-Z)} \; dF(t) \\ &= \varphi(\lambda(1-Z)). \end{aligned} \tag{2.2.32}$$

Hence, (2.2.30), (2.2.31), and (2.2.32) give the desired result.

2.3 Some Properties of the Time Between Balks

The theorems in this section refer to the random variables defined by (2.1.5) and (2.1.6). These theorems are not only useful in discussion of the queueing problem, but also provide powerful results for the inventory problem to be discussed in Chapters 5 and 6.

Theorem 2.3.1

(a) $M_1, M_2, M_3, \ldots$, are mutually independent random variables,

(b) $M_2, M_3, M_4, \ldots$, are identically distributed,

(c) $\mathcal{E}(M_j) = \mu_{KK}$, $j \geq 2$, with μ_{KK} given by (2.2.29).

Proof of Theorem 2.3.1

Let $B(i,j)$ denote the event

$$\{Q_i < K,\ Q_{i+1} < K, \ldots,\ Q_{j-1} < K,\ Q_j = K\}.$$

Let $k_1, k_2, k_3, \ldots$, be a sequence of positive integers and define $n_j = k_1 + \ldots + k_j$. Then for any $m \geq 2$,

$$\begin{aligned}
&\Pr\{M_1 = k_1,\ M_2 = k_2, \ldots,\ M_m = k_m\} \\
&= \Pr\{B(1,k_1),\ B(k_1+1,n_2), \ldots,\ B(n_{m-1}+1,n_m) \mid Q_0\} \\
&= \Pr\{B(n_{m-1}+1,n_m) \mid Q_{n_{m-1}} = K\}\ \Pr\{B(n_{m-2}+1,n_{m-1}) \mid Q_{n_{m-2}} = K\} \\
&\quad \ldots\ \Pr\{B(k_1+1,n_2) \mid Q_{k_1} = K\}\ \Pr\{B(1,k_1) \mid Q_0\} \\
&= \Pr\{B(1,k_m) \mid Q_0 = K\}\ \Pr\{B(1,k_{m-1}) \mid Q_0 = K\} \\
&\quad \ldots\ \Pr\{B(1,k_2) \mid Q_0 = K\}\ \Pr\{B(1,k_1) \mid Q_0\}\ . \qquad (2.3.1)
\end{aligned}$$

The second equality above follows by the Markov property of Q_n, and the last equality follows by time-homogeneity of Q_n. Hence,

$$\Pr\{M_1 = k_1, \ldots,\ M_m = k_m\} = \Pr\{M_1 = k_1\}\ \ldots\ \Pr\{M_m = k_m\}.$$

Therefore, the M_j are independent, and by examining the last expression in (2.3.1), we see that M_j $(j \geq 2)$ are identically distributed.

Now, for $j \geq 2$

$$\Pr\{M_j = k\} = \Pr\{Q_1 < K, \ldots, Q_{k-1} < K, Q_k = K \mid Q_0 = K\}$$

$$= \Pr\{\text{the first passage from states K to K takes k stages}\}.$$

Hence,

$$\mathcal{E}(M_j) = \text{mean recurrence time of state K}$$

$$= \mu_{KK}.$$

Applying Theorem 2.2.2, we complete the proof.

Theorem 2.3.2

(a) $V_1, V_2, V_3, \ldots$, are mutually independent random variables,

(b) $V_2, V_3, V_4, \ldots$, are identically distributed,

(c) $\mathcal{E}(V_j) = \mathcal{E}(M_j)\,\mathcal{E}(u_1), \quad j \geq 1$

$$= \mu_{KK}\,\mathcal{E}(u_1), \quad j \geq 2$$

where μ_{KK} is given by (2.2.29) and u_1 is given by (2.1.2).

Proof of Theorem 2.3.2

Since,

$$V_{j+1} = \sigma_{N_{j+1}} - \sigma_{N_j}$$

$$= u_{N_j+1} + \ldots + u_{N_{j+1}},$$

parts (a) and (b) follow directly from Theorem 2.3.1 (a) and (b), the assumed independence of the u_k, and the independence of u_{n+1} and Q_n.

Assume that $Q_0 = i$, $(1 \leq i \leq K)$. Let

$$\sigma_n^* = \sigma_n - \sigma_o$$

so that

$$\sigma_n^* = u_1 + \ldots + u_n, \quad n \geq 1.$$

Now $(\sigma_n^* - n\mathcal{E}(u_1))$ is a martingale and

$$\mathcal{E}(\sigma_n^* - n\mathcal{E}(u_1)) = 0, \text{ for all } n.$$

The event $\{M_1 > k\} \in \underline{B}_k$, where $\underline{B}_k$ is the σ-field of events generated by $\{\sigma_1^*, \ldots, \sigma_k^*\}$ and $\{w_1, \ldots, w_{k+i}\}$.

Clearly, $\underline{B}_k \subset \underline{B}_{k+1}$.

Hence, M_1 is an optional stopping rule and has no effect on the martingale property. See, for example, Feller (1966, p. 214).

Therefore,

$$\begin{aligned}\mathcal{E}(V_1 - M_1\,\mathcal{E}(u_1)) &= \mathcal{E}(\sigma_{M_1}^* - M_1\,\mathcal{E}(u_1)) \\ &= 0.\end{aligned}$$

Now let $i = K$ in the above solution so that M_1 has the same distribution as M_j $(j \geq 2)$. Then

$$\mathcal{E}(V_j - M_j\mathcal{E}(u_1)) = \mathcal{E}(V_1 - M_1\mathcal{E}(u_1)) = 0, \quad j \geq 2,$$

and part (c) follows immediately from Theorem 2.3.1(c), thus completing the proof.

2.4 The Inverse of a Special Triangular Matrix

Let $\beta_0, \beta_1, \beta_2, \ldots$, be a sequence of numbers such that $\beta_0 \neq 0$. If B is a (n+1) x (n+1) matrix of the form

$$B = \begin{bmatrix} \beta_0 & & & & & \\ \beta_1 & \beta_0 & & & & \\ \beta_2 & \beta_1 & \beta_0 & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \beta_n & \beta_{n-1} & \beta_{n-2} & \cdots & \beta_1 & \beta_0 \end{bmatrix}, \qquad (2.4.1)$$

then B^{-1} is obviously of the form

$$B^{-1} = \begin{bmatrix} \beta^{(0)} & & & & & \\ \beta^{(1)} & \beta^{(0)} & & & & \\ \beta^{(2)} & \beta^{(1)} & \beta^{(0)} & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \beta^{(n)} & \beta^{(n-1)} & \beta^{(n-2)} & \cdots & \beta^{(1)} & \beta^{(0)} \end{bmatrix}. \qquad (2.4.2)$$

We show the following:

Theorem 2.4.1

$$\beta^{(k)} = \begin{cases} 1/\beta_0, & k = 0, \\ \sum_{j=1}^{k} (-1)^j A(j:k)/\beta_0^{j+1}, & k \geq 1, \end{cases} \qquad (2.4.3)$$

with

$$A(j:k) = \sum_{i_1 + \ldots + i_j = k} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_j}.$$

$$(i_1 \geq 1, \ldots, i_j \geq 1)$$

If, in addition, $\sum_{j=0}^{\infty} \beta_j$ converges,

$$\beta^{(k)} = C_Z^{(k)} \; 1/B(Z), \quad k \geq 0 \tag{2.4.4}$$

with $B(Z) = \sum_{j=0}^{\infty} \beta_j Z^j$.

Proof of Theorem 2.4.1

It is clear that $\beta^{(0)} = 1/\beta_0$. To show (2.4.3) true for $k \geq 1$, we need simply verify that

$$\sum_{k=0}^{m} \beta^{(k)} \beta_{m-k} = 0, \quad m = 1,2,\ldots,n.$$

Now,

$$\sum_{k=0}^{m} \beta^{(k)} \beta_{m-k} = \beta_m/\beta_0 + \sum_{k=1}^{m} \beta^{(k)} \beta_{m-k}$$

$$= \beta_m/\beta_0 + \sum_{k=1}^{m} \beta_{m-k} \sum_{j=1}^{k} (-1)^j A(j:k)/\beta_0^{j+1}$$

$$= \beta_m/\beta_0 + \sum_{j=1}^{m} (-1)^j/\beta_0^{j+1} \sum_{k=j}^{m} \beta_{m-k} A(j:k). \tag{2.4.5}$$

The coefficient of $1/\beta_0$ in the last expression is $\beta_m - A(1:m) = 0$, and for $j = 1,2,\ldots,m-1$ the coefficient of

$$1/\beta_0^{j+1} = (-1)^j \left\{ \sum_{k=j}^{m-1} \beta_{m-k} A(j:k) - A(j+1:m) \right\}$$

$$= (-1)^j \left\{ \sum_{k=j}^{m-1} \sum_{i_1 + \ldots + i_j = k} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_j} \beta_{m-k} - \sum_{i_1 + \ldots + i_{j+1} = m} \beta_{i_1} \cdots \beta_{i_{j+1}} \right\} = 0.$$

Hence, (2.4.5) is zero and, therefore, (2.4.3) is true.

For $n \geq k \geq 1$,

$$A(j:k) = C_Z^{(k)} (\beta_1 Z + \beta_2 Z^2 + \ldots + \beta_n Z^n)^j.$$

Hence, from (2.4.3)

$$\beta^{(k)} = C_Z^{(k)} \sum_{j=1}^{k} [(-1)^j/\beta_0^{j+1}](\beta_1 Z + \ldots + \beta_n Z^n)^j$$

$$= (-1/\beta_0) C_Z^{(k)} \left[\frac{(\beta_1 Z + \ldots + \beta_n Z^n)}{\beta_0} \cdot \left\{ \frac{1 - [(\beta_1 Z + \ldots + \beta_n Z^n)/-\beta_0]^k}{1 + \left[\frac{\beta_1 Z + \ldots + \beta_n Z^n}{\beta_0} \right]} \right\} \right].$$

The term in brackets in the numerator of the last expression above contributes nothing to the coefficient of Z^k and hence can be dropped.

Therefore,

$$\beta^{(k)} = (-1/\beta_0)\, C_Z^{(k)} \frac{\beta_1 Z + \ldots + \beta_n Z^n}{\beta_0+\beta_1 Z+\ldots+\beta_n Z^n} .$$

But, by adding and subtracting β_0 in the numerator, we have, for $n \geq k \geq 1$,

$$\beta^{(k)} = (-1/\beta_0)\, C_Z^{(k)} \left(1 - \frac{\beta_0}{\beta_0 + \ldots + \beta_n Z^n}\right)$$

$$= C_Z^{(k)}\, 1/(\beta_0 + \ldots + \beta_n Z^n) . \qquad (2.4.6)$$

The last equality follows since $Z^0 = 1$ and we have taken $k \geq 1$.

But (2.4.6) holds for all $n \geq k$, hence

$$\beta^{(k)} = C_Z^{(k)}\, 1/B(Z), \qquad k \geq 1.$$

That (2.4.4) holds for $k = 0$ can be seen by

$$C_Z^{(0)}\, 1/B(Z) = C_Z^{(0)}\, 1/[1-(+1-B(Z))]$$

$$= C_Z^{(0)} \sum_{j=0}^{\infty} (1 - B(Z))^j$$

$$= \sum_{j=0}^{\infty} (1 - \beta_0)^j = 1/\beta_0 = \beta^{(0)} .$$

The proof is now complete.

To verify equation (2.2.31), it is noted that the matrix B above equation (2.2.31) is of the form (2.4.1) with

$$\beta_k = \begin{cases} \alpha_k, & k \neq 1, \\ \alpha_1 - 1, & k = 1, \end{cases}$$

and, therefore, $B(Z) = K(Z) - Z$.

Applying Theorem 2.4.1 with $n = K-2$ to obtain the solution of our particular B^{-1}, we have from Lemma 2.2.3

$$\xi_K - \xi_{K-1} = (1,1,\ldots,1)\, B^{-1} \begin{bmatrix} +1 \\ -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

$$= \beta^{(K-2)}$$

$$= C_Z^{(K-2)}\, 1/(K(Z) - Z).$$

CHAPTER 3

THE QUEUE GI/M/∞ WITH BALKING AT QUEUES OF LENGTH K-1

3.1 The Basic System

Consider a queueing system in which customers arrive in the system at times, $\ldots,\sigma_{-2},\sigma_{-1},\sigma_0,\sigma_1,\sigma_2,\ldots$, such that the inter-arrival times

$$u_j = \sigma_j - \sigma_{j-1}, \qquad j \geq 1, \tag{3.1.1}$$

are mutually independent. The distribution function of u_j will be denoted by

$$\Pr\{u_j \leq u\} = F(u), \qquad u \geq 0, \quad j \geq 1. \tag{3.1.2}$$

We assume there is a sufficient number of servers so that, if a person joins the queue, his service commences immediately. The queue length at any time t is the number of persons being served at time t (no one has to wait for service) or, equivalently, the number of busy servers at time t. Since a customer balks at a queue of length K-1, there are never more than K-1 servers busy at any one time. Hence, the queues GI/M/∞ and GI/M/s, for $s \geq K-1$, both with balking at queues of length K-1, are identical.

We also have the apparent absurdity that a person would balk from a system with an infinite number of servers. It would be better

in this case to assert that the customer, who arrives to find K-1 servers busy, is rejected by the system.

The successive service times for customers who join the queue are denoted by $w_1, w_2, w_3, \ldots$, and are assumed to be mutually independent random variables. Any w_j is also assumed to be independent of the arrival times. The distribution function of w_j is assumed to be

$$\Pr\{w_j \leq w\} = 1 - e^{-\lambda w}, \qquad w \geq 0, \quad j \geq 1. \tag{3.1.3}$$

As before, we let $\{Q(t); -\infty < t < +\infty\}$ be the stochastic process such that $Q(t)$ represents the number of people in the system at time t. The number of people that can be in the system at any one time is restricted to K by requiring $Q(\sigma + 0) = K-1$ whenever $Q(\sigma) = K$. See Section 2.1 for a more thorough discussion of $Q(t)$.

We are interested in the development of $Q(t)$ beyond the time point σ_0. Without loss of generality, σ_0 could be taken to be zero.

Once again we are interested in the random variables

$$N_1 = \inf \{k > 0 \,|\, Q(\sigma_k) = K\},$$

$$N_n = \inf \{k > N_{n-1} \,|\, Q(\sigma_k) = K\}, \qquad n \geq 2, \tag{3.1.4}$$

$$M_1 = N_1,$$

$$M_j = N_j - N_{j-1}, \qquad j \geq 2, \tag{3.1.5}$$

and

$$V_1 = V_{N_1} - \sigma_0,$$

$$V_j = \sigma_{N_j} - \sigma_{N_{j-1}}, \qquad j \geq 2. \tag{3.1.6}$$

A complete description of these random variables is given in Section 2.1.

Since the service times are negative exponentially distributed, we find that many of the results derived in Chapter 2 will apply to the queue $GI/M/\infty$ with balking at queues of length K-1 without any change in the proofs.

As before, we follow a systematic approach to find the solution of $\mathcal{E}(V_j)$.

3.2 An Imbedded Markov Chain

Lemma 2.2.2 applies to the stochastic process $\{Q(t);\ -\infty < t < +\infty\}$ defined in Section 3.1 and Q(t) is, therefore, in general, a non-Markovian process. However, there exists an imbedded Markov chain defined by

$$Q_n = Q(\sigma_n), \qquad n = 0,1,2,.... \tag{3.2.1}$$

Figures 3.1 and 3.2 give the relation between Q(t) and Q_n. Q_n clearly represents the number of persons in the system at the instant the n-th customer arrives. Once again we shall restrict our attention to the cases $K \geq 2$, for when $K = 1$, $\Pr\{Q_n = 1\} = 1$, $n = 0,1,2,....$

Information obtained from the stochastic process $\{Q_n;\ n = 0,1,2,...\}$ will provide sufficient information about $\{Q(t);\ -\infty < t < +\infty\}$ for our purposes.

Theorem 3.2.1

The stochastic process $\{Q_n;\ n = 0,1,2,...\}$ defined by (3.2.1) has the following properties:

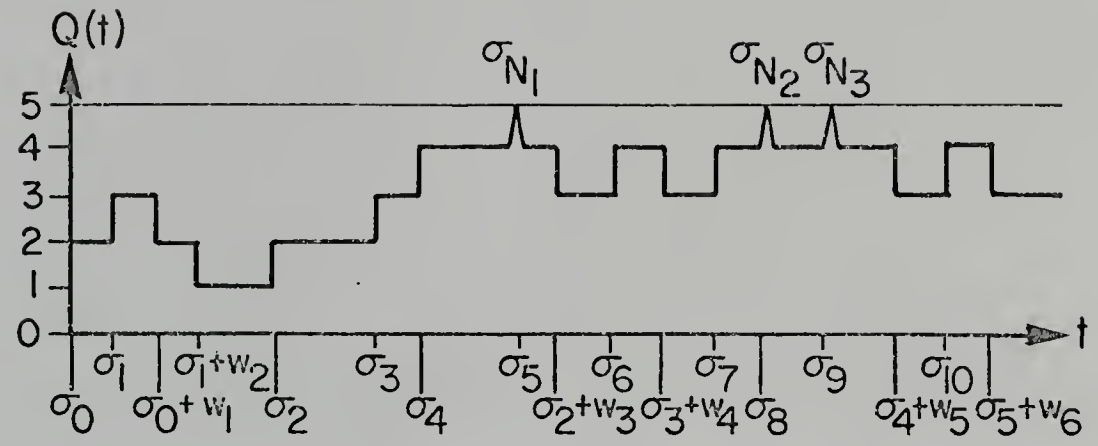

Figure 3.1. A Typical Path of Q(t) for GI/M/∞ with Balking at Queues of Length 4.

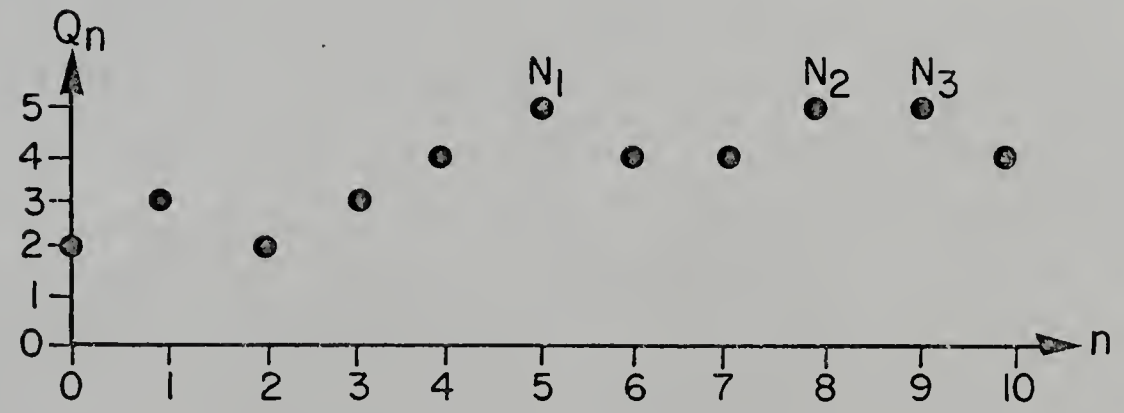

Figure 3.2. Path of Q_n Corresponding to Q(t) in Figure 3.1.

(a) Q_n is a Markov chain;

(b) Q_n is time-homogeneous;

(c) The class $\{1,2,\ldots,K\}$ of states on which Q_n is defined is an aperiodic, positive persistent communicating class; and

(d) The one-step transition probability matrix P is given by

↗	K	K-1	...	3	2	1
K	b(K-1,0)	b(K-1,1)	...	b(K-1,K-3)	b(K-1,K-2)	b(K-1,K-1)
K-1	b(K-1,0)	b(K-1,1)	...	b(K-1,K-3)	b(K-1,K-2)	b(K-1,K-1)
K-2	0	b(K-2,0)	...	b(K-2,K-4)	b(K-2,K-3)	b(K-2,K-2)
.	.	.		.	.	.
.	.	.		.	.	.
.	.	.		.	.	.
3	0	0	...	b(3,1)	b(3,2)	b(3,3)
2	0	0	...	b(2,0)	b(2,1)	b(2,2)
1	0	0	...	0	b(1,0)	b(1,1)

where

$$b(n,k) = \int_0^\infty \binom{n}{k} (1 - e^{-\lambda u})^k (e^{-\lambda u})^{n-k} \, dF(u). \qquad (3.2.2)$$

Proof of Theorem 3.2.1

The proofs of parts (a) and (c) are identical to those given for Theorem 2.2.1 (a) and (c). We need only show parts (b) and (d).

Let $U(\cdot)$ be the unit step function at zero. Let X_{n+1} be the number of customers who complete their service in $(\sigma_n, \sigma_{n+1}]$. Then

$$Q_{n+1} = Q_n + 1 - X_{n+1}, \qquad \text{if } Q_n < K,$$

$$Q_{n+1} = Q_n + 1 - X_{n+1} - 1, \qquad \text{if } Q_n = K,$$

so that

$$Q_{n+1} = Q_n + 1 - X_{n+1} - U(Q_n - K), \quad Q_n \le K.$$

Since we are looking at $Q(t)$ at successive arrival times, we have

$$1 \le Q_{n+1} \le Q_n + 1.$$

By the balking aspect of the problem, $Q_n \le K$.

Hence,

$$\Pr\{Q_{n+1} = k | Q_n = j\} = 0, \quad \text{for } k > j + 1. \tag{3.2.3}$$

Further,

$$\begin{aligned}\Pr\{Q_{n+1} = k | Q_n = K\} &= \Pr\{K - X_{n+1} - 1 + 1 = k | Q_n = K\} \\ &= \Pr\{(K-1) - X_{n+1} + 1 = k | Q_n = K-1\} \\ &= \Pr\{Q_{n+1} = k | Q_n = K-1\}. \end{aligned} \tag{3.2.4}$$

For $1 \le j \le K-1$ and $k \le j+1$, we have

$$\begin{aligned}\Pr\{Q_{n+1} = k | Q_n = j\} &= \Pr\{j - X_{n+1} + 1 = k | Q_n = j\} \\ &= \Pr\{X_{n+1} = j + 1 - k | Q_n = j\} \\ &= \Pr\{\text{exactly } j + 1 - k \text{ persons out} \\ &\qquad \text{of } j \text{ complete their service in} \\ &\qquad (\sigma_n, \sigma_{n+1}]\} \\ &= \mathop{\mathcal{E}}_{u_{n+1}} (\Pr\{\text{exactly } j+1 - k \text{ independent events} \\ &\qquad \{w_1 \le u_{n+1}\} \text{ occur out of } j \text{ possibilities}\}) \\ &= b(j, j + 1 - k). \end{aligned} \tag{3.2.5}$$

Equations (3.2.3) through (3.2.5) are independent of n and hence part (b) follows. Application of these equations when j,k = 1,2,...,K gives the matrix P, thus completing the proof.

Let μ_{KK} be the mean recurrence time of the state K for the imbedded Markov chain of Theorem 3.2.1. If we let $p_{n,k} = b(n,k-1)$, we have that the matrix P^* of Corollary 2.2.1 and the matrix P of Theorem 3.2.1 are identical. Hence, $\mu_{22},\mu_{33},\mu_{44},\ldots$, satisfy

$$\mu_{KK} = \begin{cases} 1/b(1,0), & K = 2, \\ 1/b(K-1,0)\left[1 + \sum_{k=2}^{K-1} b(K-1,K-k+1) \sum_{j=k}^{K-1} \mu_{jj}\right], & K \geq 3. \end{cases} \tag{3.2.6}$$

The matrix P of Theorem 3.2.1 and equation (3.2.6) both contain the quantity b(n,k) defined by equation (3.2.2). However, this integral expression is not a form that lends itself to easy evaluation. Fortunately, b(n,k) can be expressed as a function of the parameter λ of (3.1.3) and the Laplace transform of (3.1.2) in the following manner. Let

$$\varphi(\theta) = \int_0^\infty e^{-\theta u}\, dF(u), \qquad \theta \geq 0, \tag{3.2.7}$$

then

$$\begin{aligned} b(n,k) &= \int_0^\infty \binom{n}{k} (1 - e^{-\lambda u})^k (e^{-\lambda u})^{n-k}\, dF(u) \\ &= \binom{n}{k} \int_0^\infty \sum_{j=0}^{k} \binom{k}{j} (-e^{-\lambda u})^j (e^{-\lambda u})^{n-k}\, dF(u) \\ &= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_0^\infty e^{-(n-k+j)\lambda u}\, dF(u) \\ &= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j \varphi(\lambda(n-k+j)). \end{aligned} \tag{3.2.8}$$

3.3 Some Properties of the Time Between Balks

The following theorems refer to the random variables defined by (3.1.5) and (3.1.6). The proofs of these theorems are identical to those given for Theorems 2.3.1 and 2.3.2. Hence, only the statements of the theorems will be given.

Theorem 3.3.1

(a) $M_1, M_2, M_3, \ldots$, are mutually independent random variables,

(b) $M_2, M_3, M_4, \ldots$, are identically distributed,

(c) $\mathcal{E}(M_j) = \mu_{KK}, \quad j \geq 2$

with μ_{KK} given by (3.2.6).

Theorem 3.3.2

(a) $V_1, V_2, V_3, \ldots$, are mutually independent random variables,

(b) $V_2, V_3, V_4, \ldots$, are identically distributed,

(c) $\mathcal{E}(V_j) = \mathcal{E}(M_j)\,\mathcal{E}(u_1), \quad j \geq 1$

$= \mu_{KK}\,\mathcal{E}(u_1), \quad j \geq 2$

with μ_{KK} given by (3.2.6) and u_1 defined by (3.1.2).

CHAPTER 4

THE QUEUE GI/D/1 WITH BALKING AT QUEUES OF LENGTH K-1

4.1 The Basic System

Consider a queueing system in which customers arrive in the system at times, $\ldots,\sigma_{-2},\sigma_{-1},\sigma_0,\sigma_1,\sigma_2,\ldots$, such that the inter-arrival times

$$u_j = \sigma_j - \sigma_{j-1}, \qquad j \geq 1, \tag{4.1.1}$$

are mutually independent. The distribution function of u_j will be denoted by

$$\Pr\{u_j \leq u\} = F(u), \quad u \geq 0, \quad j \geq 1. \tag{4.1.2}$$

One server is available to handle the needs of the customers. This server dispenses his service on a strict "first come, first served" basis. The service time of any customer who joins the queue is assumed to be a constant value b.

Let $\{Q(t); -\infty < t < +\infty\}$ be the stochastic process such that $Q(t)$ represents the number of people in the system at time t. The number of people that can be in the system at any one time is restricted to K by requiring $Q(\sigma + 0) = K-1$ whenever $Q(\sigma) = K$. A more thorough discussion of $Q(t)$ and its relation to the queue length is found in Section 2.1. As in the previous two chapters,

we are interested in the development of $Q(t)$ beyond the time point σ_0. We could, therefore, take $\sigma_0 = 0$ without loss of generality.

Define

$$N_1 = \inf \{k > 0 | Q(\sigma_k) = K\},$$
$$N_n = \inf \{k > N_{n-1} | Q(\sigma_k) = K\}, \quad n \geq 2, \tag{4.1.3}$$

$$M_1 = N_1,$$
$$M_j = N_j - N_{j-1}, \quad j \geq 2, \tag{4.1.4}$$

and

$$V_1 = \sigma_{N_1} - \sigma_0,$$
$$V_j = \sigma_{N_j} - \sigma_{N_{j-1}}, \quad j \geq 2. \tag{4.1.5}$$

A complete description of these random variables is given in Section 2.1.

Our ultimate objective is to find an expression for $\mathcal{E}(V_j)$. Unfortunately, we are only able to obtain an exact expression for $\mathcal{E}(V_j)$ in terms of quantities that are difficult (if not impossible) to obtain. The results that we do establish are based on the concept of the waiting time in the system. The definition of the waiting time and our motivation for its use in the search for a solution to $\mathcal{E}(V_j)$ now follow.

Let $W(t)$ be the amount of time it would take our server to finish serving all of the customers present in the queue at time t. $W(t)$ is then called the waiting time in the system at time t. If σ is the time of an arrival into the system, then

$$Q(\sigma) < K \quad \text{implies} \quad W(\sigma) = W(\sigma-0) + b$$

and

$$Q(\sigma) = K \quad \text{implies} \quad W(\sigma) = W(\sigma-0).$$

The latter condition reflects the fact that a customer arriving in the system to find K-1 persons already in the queue, leaves without waiting to be served.

By the balking aspect of the problem, at most K-1 persons may be in the queue at any particular time t. Since the service time is a constant, b, we have $0 \leq W(t) \leq (K-1)b$.

Again let σ represent the time of an arrival into the system. Clearly,

$$Q(\sigma) = 1 \quad \text{if and only if} \quad W(\sigma-0) = 0.$$

Further, if $Q(\sigma) = j$ $(j = 2,3,\ldots,K)$ we must have $Q(\sigma-0) = j-1$. Since the service time for any one customer is b, a constant,

$$(j-2)b < W(\sigma-0) \leq (j-1)b.$$

That is, for $j = 2,3,\ldots,K$

$$Q(\sigma) = j \quad \text{if and only if} \quad (j-2) < W(\sigma-0) \leq (j-1)b.$$

Hence, complete knowledge of the stochastic process defined by

$$Q_n = Q(\sigma_n), \qquad n = 0,1,2,\ldots,$$

can be obtained from a knowledge of the probability law of the stochastic process defined by

$$W_n = W(\sigma_n-0), \quad n = 0,1,2,\ldots. \tag{4.1.6}$$

Figures 4.1 and 4.2 give typical realizations of $Q(t)$ and W_n.

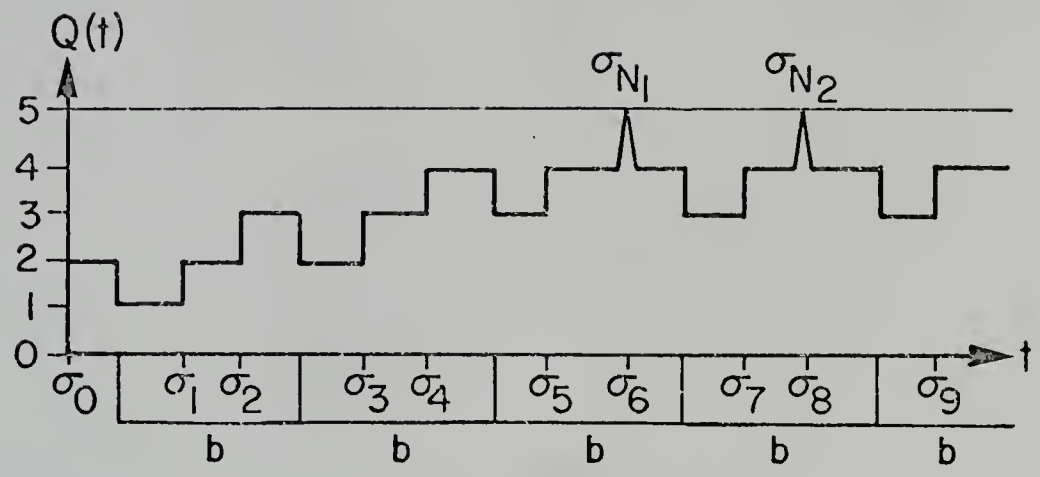

Figure 4.1. A Typical Path of Q(t) for GI/D/1 with Balking at Queues of Length 4.

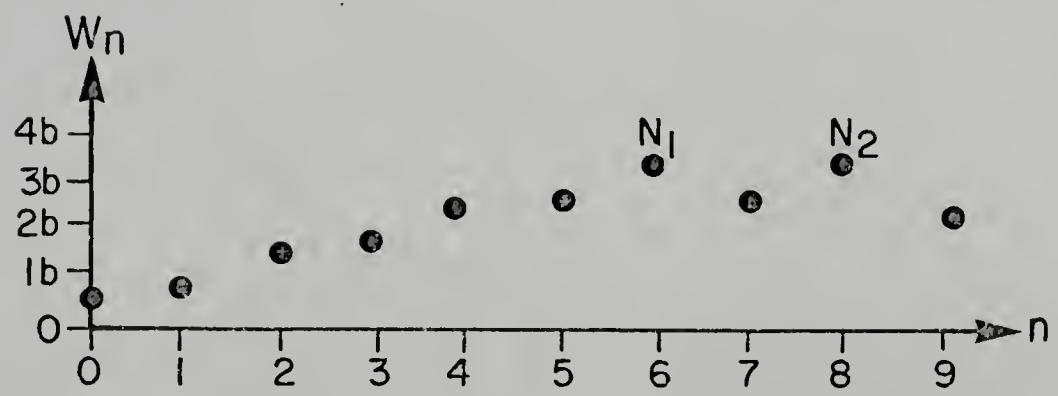

Figure 4.2. Path of W_n Corresponding to Q(t) in Figure 4.1.

The random variables defined by (4.1.3) can now be expressed in the equivalent form

$$N_1 = \inf \{k > 0 | W_k > (K-2)b\},$$
$$N_n = \inf \{k > N_{n-1} | W_k > (K-2)b\}, \quad n \geq 2. \tag{4.1.7}$$

It may be further shown, by considering a slight modification of the proof of Lemma 2.2.2, that both $Q(t)$ and $W(t)$ are non-Markovian processes. Since we now have constant rather than negative exponential service times, it is also true that Q_n is non-Markovian. However, it will be shown in the next section that W_n is a Markov process. Because of the Markov nature of W_n and the equivalence of (4.1.3) and (4.1.7), we are led to consider the stochastic process $\{W_n; n = 0,1,2,\ldots\}$ in our search for an expression for $\mathcal{E}(V_j)$.

As always, we shall ignore the trivial case when $K = 1$.

4.2 The Waiting Time in the System

Let $\{W_n; n = 0,1,2,\ldots\}$ be the stochastic process defined by (4.1.6) so that W_n is the waiting time in the system immediately preceding the n-th arrival into the system. Then it is clear that $W_n \leq (K-2)b$, i.e., $Q_n < K$, implies

$$W_{n+1} = \begin{cases} 0, & \text{if } W_n + b \leq u_{n+1}, \\ W_n + b - u_{n+1}, & \text{if } W_n + b > u_{n+1}, \end{cases}$$

and $W_n > (K-2)b$, i.e., $Q_n = K$, implies

$$W_{n+1} = \begin{cases} 0, & \text{if } W_n \leq u_{n+1}, \\ W_n - u_{n+1}, & \text{if } W_n > u_{n+1}. \end{cases}$$

If $U(\cdot)$ is the unit step function at zero, we can rewrite the above expressions in the form

$$W_{n+1} = \max \{0, W_n - u_{n+1} + b\, U((K-2)b - W_n)\}, \quad n \geq 0. \tag{4.2.1}$$

We now note the relation between (4.2.1) and the analogous expression for the waiting time just prior to an arrival in the queue GI/D/1 (no balking). If we let $\{W_n^*; n = 0,1,2,\ldots\}$ be the stochastic process that represents the latter waiting time, we have the well-known result

$$W_{n+1}^* = \max \{0, W_n^* - u_{n+1} + b\}, \quad n \geq 0. \tag{4.2.2}$$

See, for example, Prabhu (1965b).

The difference between W_n and W_n^* is that, for the former, a person who enters the system and faces a queue of length K-1 balks and adds no service time to the system.

We now formally state and prove some basic properties of the stochastic process $\{W_n; n = 0,1,2,\ldots\}$.

Theorem 4.2.1

The stochastic process $\{W_n; n = 0,1,2,\ldots\}$ is a time-homogeneous Markov process concentrated on the continuous state space $[0, (K-1)b]$ with one-step transition distribution

$$\Pr\{W_{n+1} \leq x | W_n = y\} = \begin{cases} 1 - F(y-x-0), & y > (K-2)b, \\ 1 - F(y+b-x-0), & y \leq (K-2)b. \end{cases} \tag{4.2.3}$$

Note that the interval on which (4.2.3) is concentrated is actually a sub-interval of $[0, (K-1)b]$, this sub-interval being a function of y.

Proof of Theorem 4.2.1

Let $m_1, m_2, m_3, \ldots$, be a sequence of integers such that $m_1 < m_2 < \ldots < m_k < n$. Then

$$\Pr\{W_{n+1} \le x | W_n = y, W_{m_k} = y_k, \ldots, W_{m_1} = y_1\}$$

$$= \Pr\{\max\{y - u_{n+1} + bU((K-2)b-y), 0\} \le x\}$$

$$= \Pr\{W_{n+1} \le x | W_n = y\}.$$

Hence W_n is Markovian. Now,

$$\Pr\{W_{n+1} \le x | W_n = y\} = \Pr\{W_{n+1} = 0 | W_n = y\}$$

$$+ \Pr\{0 < W_{n+1} \le x | W_n = y\}$$

$$= \Pr\{y - u_{n+1} + bU((K-2)b-y) \le 0\}$$

$$+ \Pr\{0 < y - u_{n+1} + bU((K-2)b-y) \le x\}$$

$$= \Pr\{u_{n+1} \ge y + bU((K-2)b-y) - x\}$$

$$= 1 - F(y + bU((K-2)b - y) - x - 0), \qquad (4.2.4)$$

which is independent of n, and therefore W_n is time-homogeneous. Equation (4.2.4) is the same as (4.2.3), thus completing the proof.

To simplify the notation, we write

$$P_n(y, x) = \Pr\{W_n \le x | W_0 = y\}, \quad n \ge 1, \qquad (4.2.5)$$

and

$$P(y, x) = P_1(y, x). \qquad (4.2.6)$$

Theorem 4.2.2

The n-step transition distribution functions defined by (4.2.5) are concentrated on the continuous state space $[0, (K-1)b]$ (or a subinterval of it) and satisfy

$$P_{n+1}(z,x) = \int_{\max(x,(K-2)b)}^{\infty} P_n(z,y)\, d_y F(y-x-0)$$

$$+ \int_{\max(0-,x-b)}^{(K-2)b} P_n(z,y)\, d_y F(y+b-x-0)$$

$$- P_n(z,(K-2)b)\,[F((K-1)b-x-0)$$

$$- F((K-2)b-x-0)]. \qquad (4.2.7)$$

Proof of Theorem 4.2.2

By the Chapman-Kolmogorov equations,

$$P_{n+1}(z,x) = \int_{0-}^{(K-1)b} P(y,x)\, P_n(z,dy). \qquad (4.2.8)$$

Integrating by parts, (4.2.8) becomes

$$P_{n+1}(z,x) = [P(y,x)\, P_n(z,y)]_{y=0-}^{y=(K-1)b}$$

$$- \int_{0-}^{(K-1)b} P_n(z,y)\, P(dy,x)$$

$$= P((K-1)b,x) - \int_{0-}^{(K-1)b} P_n(z,y)\, P(dy,x).$$

$$(4.2.9)$$

By Theorem 4.2.1, we have

$$P(dy,x) = \begin{cases} 0, & y \le x, \\ -d_y F(y-x-0), & y > x, \end{cases} \qquad (K-2)b < y \le (K-1)b,$$

$$P(dy,x) = \begin{cases} 0, & y \le x-b, \\ -d_y F(y+b-x-0), & y > x-b, \end{cases} \qquad 0 \le y < (K-2)b,$$

$$P(dy,x) = F((K-1)b-x-0) - F((K-2)b-x-0), \qquad y = (K-2)b,$$

so that (4.2.9) becomes

$$\begin{aligned} P_{n+1}(z,x) = {} & 1 - F((K-1)b-x-0) \\ & + \int_{\max(x,(K-2)b)}^{(K-1)b} P_n(z,y)\, d_y F(y-x-0) \\ & + \int_{\max(0-,x-b)}^{(K-2)b} P_n(z,y)\, d_y F(y+b-x-0) \\ & - P_n(z,(K-2)b)[F((K-1)b-x-0) \\ & \qquad - F((K-2)b-x-0)] . \end{aligned} \qquad (4.2.10)$$

But $P_n(z,y) = 1$, for $y > (K-1)b$. Hence (4.2.10) becomes (4.2.7), thus completing the proof.

Various attempts have been made to establish the stationary distribution of W_n, all without success. Since the stochastic kernal $P(y,x)$ does not satisfy the regularity conditions stated in Feller (1966, Sec. VIII, 7), we are not even sure if W_n possesses a stationary distribution.

As will be seen in the next section, the most important result of this section is the Markov property of W_n established in Theorem 4.2.1.

4.3 Some Properties of the Time Between Balks

We are now ready to obtain solutions for the expected values of the random variables M_j and V_j defined by (4.1.4) and (4.1.5), respectively. In the previous chapters exact results were derived for these expectations, but in this chapter we must be content to utilize unsolved expressions for the expectations of interest. In order to reach our objectives, we make use of the properties of the following quantities.

Let

$$S_n = nb - u_1 - \ldots - u_n, \qquad n \geq 1,$$

and, for $0 \leq y \leq (K-1)b$, y a real number, let

$$y' = \begin{cases} y, & y \leq (K-2)b, \\ y - b, & y > (K-2)b. \end{cases}$$

Define

$$N(y) = \inf \{n > 0 | W_n > (K-2)b;\ W_0 = y\} \tag{4.3.1}$$

(so that $N(y)$ represents the number of arrivals until a balk occurs, conditional on an initial amount y of waiting time in the system), and

$$M(y) = \inf \{n > 0 | W_n = 0 \text{ or } W_n > (K-2)b;\ W_0 = y\} \tag{4.3.2}$$

(so that $M(y)$ is the number of arrivals until a customer either enters an empty queue or balks, conditional on an initial amount y of waiting time in the system).

It is clear that $M(y)$ has the equivalent representation

$$M(y) = \inf \{n > 0 | S_n \leq -y' \text{ or } S_n > (K-2)b - y'\}$$

and, hence, is the index at which the random walk $\{S_n;\ n = 1,2,3,\ldots\}$ first leaves the interval $(-y', (K-2)b-y']$.

Finally, let $J = J(y)$ be the random variable that represents the number of customers who enter an empty queue prior to the first person to balk, conditional on an initial amount y of waiting time in the system. That is, J is the number of times the waiting time process W_n $(n \geq 1)$ takes the value zero before it takes a value greater than $(K-2)b$

First we shall prove a few lemmas that lead to a theorem which expresses $\mathcal{E}(N(y))$ in terms of expectations and probabilities associated with the random variables $M(y)$, $S_{M(y)}$, $M(0)$, and $S_{M(0)}$. Note that when $K= 2$, $M(0) = M(y) = 1$ and $S_{M(0)} = S_{M(y)} = b-u_1$.

Lemma 4.3.1

$$\Pr\{J = 0\} = \Pr\{S_{M(y)} > (K-2)b-y'\},$$

$$\Pr\{J = j\} = \Pr\{S_{M(y)} \leq -y'\} \Pr\{S_{M(0)} \leq 0\}^{j-1} \cdot \Pr\{S_{M(0)} > (K-2)b\}, \quad j \geq 1.$$

Proof of Lemma 4.3.1

Define

$L_1 = M(y)$,

$L_j = \inf\{n > L_{j-1} | W_n = 0 \text{ or } W_n > (K-2)b;\ W_0 = y\}, \quad j \geq 2$

(so that L_j is the index of the j-th person to balk or enter an empty queue). Keep in mind that L_j is a function of y.

Now, if $j = 0$,

$$\Pr\{J = 0\} = \sum_{n=1}^{\infty} \Pr\{J = 0, N(y) = n\}$$

$$= \sum_{n=1}^{\infty} \Pr\{0 < W_1 \le (K-2)b, \ldots, 0 < W_{n-1} \le (K-2)b, W_n > (K-2)b \mid W_0 = y\}$$

$$= \sum_{n=1}^{\infty} \Pr\{0 < S_1 + y' \le (K-2)b, \ldots, 0 < S_{n-1} + y' \le (K-2)b, S_n + y' > (K-2)b\}$$

$$= \sum_{n=1}^{\infty} \Pr\{M(y) = n, S_{M(y)} + y' > (K-2)b\}$$

$$= \Pr\{S_{M(y)} > (K-2)b - y'\}.$$

For $j \ge 1$,

$$\Pr\{J = j\} = \Pr\{W_{L_1} = 0, W_{L_2} = 0, \ldots, W_{L_j} = 0, W_{L_{j+1}} > (K-2)b \mid W_0 = y\}$$

$$= \Pr\{W_{L_{j+1}} > (K-2)b \mid W_{L_j} = 0\} \Pr\{W_{L_j} = 0 \mid W_{L_{j-1}} = 0\}$$

$$\ldots \Pr\{W_{L_2} = 0 \mid W_{L_1} = 0\} \Pr\{W_{L_1} = 0 \mid W_0 = y\}, \qquad (4.3.3)$$

the last equality following from the Markov nature of W_n. But we have,

$$\Pr\{W_{L_1} = 0 \mid W_0 = y\} = \Pr\{W_{M(y)} = 0 \mid W_0 = y\}$$

$$= \sum_{n=1}^{\infty} \Pr\{W_{M(y)} = 0, M(y) = n \mid W_0 = y\}$$

$$= \sum_{n=1}^{\infty} \Pr\{S_{M(y)} + y' \le 0, M(y) = n\}$$

$$= \Pr\{S_{M(y)} \le -y'\}, \qquad (4.3.4)$$

and

$$\Pr\{W_{L_{i+1}} = 0 | W_{L_i} = 0\} = \sum_{n=1}^{\infty} \Pr\{L_{i+1} - L_i = n,\ W_{L_{i+1}} = 0 | W_{L_i} = 0\}$$

$$= \sum_{n=1}^{\infty} \Pr\{0 < W_{L_i+1} \le (K-2)b, \ldots, 0 < W_{L_i+n-1} \le (K-2)b,\ W_{L_i+n} = 0 | W_{L_i} = 0\}$$

$$= \sum_{n=1}^{\infty} \Pr\{0 < S_1 \le (K-2)b, \ldots, 0 < S_{n-1} \le (k-2)b,\ S_n \le 0\}$$

$$= \sum_{n=1}^{\infty} \Pr\{M(0) = n,\ S_{M(0)} \le 0\}$$

$$= \Pr\{S_{M(0)} \le 0\}, \qquad 1 \le i \le j-1. \tag{4.3.5}$$

Following a proof analogous to that used to obtain (4.3.5), we get

$$\Pr\{W_{L_{j+1}} > (K-2)b | W_{L_j} = 0\} = \Pr\{S_{M(0)} > (K-2)b\}. \tag{4.3.6}$$

Applying (4.3.4) through (4.3.6) to (4.3.3), we complete the proof.

Lemma 4.3.2

$$\mathcal{E}(N(y) | J=0) = \mathcal{E}(M(y) | S_{M(y)} > (K-2)b-y'),$$

$$\mathcal{E}(N(y) | J=j) = \mathcal{E}(M(y) | S_{M(y)} \le -y')$$

$$+ (j-1)\,\mathcal{E}(M(0) | S_{M(0)} \le 0)$$

$$+ \mathcal{E}(M(0) | S_{M(0)} > (K-2)b), \qquad j \ge 1.$$

Proof of Lemma 4.3.2

$$\Pr\{N(y) = n \mid J = 0\} = \Pr\{N(y) = n,\ J = 0\}/\Pr\{J = 0\}$$

$$= \Pr\{0 < S_1 + y' \le (K-2)b, \ldots, 0 < S_{n-1} + y' \le (K-2)b,$$
$$S_n + y' > (K-2)b\}/\Pr\{J = 0\}$$

$$= \Pr\{M(y) = n,\ S_{M(y)} + y' > (K-2)b\}/\Pr\{S_{M(y)} + y' > (K-2)b\}$$

$$= \Pr\{M(y) = n \mid S_{M(y)} > (K-2)b-y'\},$$

and therefore

$$\mathcal{E}(N(y) \mid J = 0) = \mathcal{E}(M(y) \mid S_{M(y)} > (K-2)b-y').$$

For $j \ge 1$, note that

$$\mathcal{E}(N(y) \mid J = j) = \mathcal{E}(L_{j+1} \mid J = j)$$

$$= \mathcal{E}(L_1 \mid J = j) + \mathcal{E}(L_2 - L_1 \mid J = j) + \ldots +$$

$$\mathcal{E}(L_{j+1} - L_j \mid J = j). \qquad (4.3.7)$$

We now solve for the various terms in the last expression of equation (4.3.7).

$$\mathcal{E}(L_1 \mid J = j) = \sum_{n=1}^{\infty} n \Pr\{J = j,\ L_1 = n\}/\Pr\{J = j\}$$

$$= \sum_{n=1}^{\infty} \frac{n \Pr\{W_{L_1}=0, \ldots, W_{L_j}=0, W_{L_{j+1}} > (K-2)b, L_1 = n \mid W_0 = y\}}{\Pr\{J = j\}}$$

$$= \sum_{n=1}^{\infty} n \Pr\{W_{L_{j+1}} > (K-2)b \mid W_{L_j} = 0\} \Pr\{W_{L_j} = 0 \mid W_{L_{j-1}} = 0\}$$

$$\ldots \Pr\{W_{L_2} = 0 \mid W_{L_1} = 0\} \frac{\Pr\{W_{L_1} = 0,\ L_1 = n \mid W_0 = y\}}{\Pr\{J = j\}}$$

$$= \sum_{n=1}^{\infty} n \Pr\{W_{L_1} = 0,\ L_1 = n \mid W_0 = y\}/\Pr\{W_{L_1} = 0 \mid W_0 = y\}$$

$$= \sum_{n=1}^{\infty} n \Pr\{M(y) = n | S_{M(y)} + y' \leq 0\}$$

$$= \mathcal{E}(M(y) | S_{M(y)} \leq - y'). \qquad (4.3.8)$$

The second equality from the last follows from (4.3.3) and the next to the last equality follows from (4.3.4). Using similar techniques as those employed in deriving (4.3.8), we have

$$\mathcal{E}(L_{i+1} - L_i | J = j) = \mathcal{E}(M(0) | S_{M(0)} \leq 0), \quad 1 \leq i \leq j-1, \qquad (4.3.9)$$

and

$$\mathcal{E}(L_{j+1} - L_j | J = j) = \mathcal{E}(M(0) | S_{M(0)} > (K-2)b). \qquad (4.3.10)$$

Applying (4.3.8) through (4.3.10) to (4.3.7), we complete the proof.

We are now ready to find the expression for $\mathcal{E}(N(y))$.

Theorem 4.3.1

If $K \geq 2$, then

$$\mathcal{E}(N(y)) = \mathcal{E}(M(y)) + \mathcal{E}(M(0)) \frac{\Pr\{S_{M(y)} \leq - y'\}}{\Pr\{S_{M(0)} > (K-2)b\}},$$

and, in particular, if $K = 2$, then

$$\mathcal{E}(N(y) = 1 + [1 - F(b+y'-0)]/F(b-0).$$

Proof of Theorem 4.3.1

For $K \geq 2$, let $A(y)$ be the event $\{S_{M(y)} + y' > (K-2)b\}$ and $A'(y)$ the complement event $\{S_{M(y)} + y' \leq 0\}$. Further, let $p(y) = \Pr\{A(y)\}$ and $q(y) = \Pr\{A'(y)\} = 1-p(y)$. We have, by Lemmas 4.3.1 and 4.3.2, that

$$\mathcal{E}(N(y)) = \sum_{j=0}^{\infty} \mathcal{E}(N(y)|J = j)\ \Pr\{J = j\}$$

$$= \mathcal{E}(M(y)|A(y))\ p(y)$$

$$+ \mathcal{E}(M(y)|A'(y))\ q(y) \sum_{j=1}^{\infty} q(0)^{j-1}\ p(0)$$

$$+ \mathcal{E}(M(0)|A'(0))\ q(y) \sum_{j=1}^{\infty} (j-1)\ q(0)^{j-1}\ p(0)$$

$$+ \mathcal{E}(M(0)|A(0))\ q(y) \sum_{j=1}^{\infty} q(0)^{j-1}\ p(0)$$

$$= \mathcal{E}(M(y)) + \mathcal{E}(M(0)|A'(0))\ q(y)\ [1/p(0) - 1]$$

$$+ \mathcal{E}(M(0)|A(0))\ q(y)$$

$$= \mathcal{E}(M(y)) + q(y)\ [\mathcal{E}(M(0)|A'(0))\ (q(0)/p(0))$$

$$+ \mathcal{E}(M(0)|A(0))\ (p(0)/p(0))]$$

$$= \mathcal{E}(M(y)) + \mathcal{E}(M(0))\ q(y)/p(0),$$

and the first half of the theorem is proved.

For K = 2, we have immediately that

$$\Pr\{N(y) = 1\} = \Pr\{W_1 > 0|W_0 = y\}$$

$$= \Pr\{S_1 + y' > 0\}$$

$$= \Pr\{u_1 < b + y'\}$$

$$= F(b+y'-0)$$

$$= \Pr\{J = 0\} \qquad (4.3.11)$$

and

$$\Pr\{N(y) = n\} = \Pr\{W_1 = 0, \ldots, W_{n-1} = 0, W_n > 0|W_0 = y\}$$

$$= \Pr\{b-u_1+y' \le 0, b-u_2 \le 0, \ldots, b-u_{n-1} \le 0,\ b-u_n > 0\}$$

$$= [1 - F(b+y'-0)]\ [1 - F(b-0)]^{n-2}\ F(b-0)$$

$$= \Pr\{J = n-1\}, \quad n \ge 2. \qquad (4.3.12)$$

From (4.3.11) and (4.3.12), $\mathcal{E}(N(y))$ for the case $K=2$ follows trivially, thus completing the proof.

We now proceed to find an approximate expression for $\mathcal{E}(N(y))$ when $K \geq 3$. Since $M(y)$ represents the index of first passage of the random walk $\{S_n;\ n = 1,2,3,\ldots\}$ out of the interval $(-y', (K-2)b-y']$, we take $y>0$ (and hence $y'>0$) so that we can use Wald's approximation to yield the following results for the random variables $M(y)$ and $S_{M(y)}$. See, for example, Ferguson (1967).

Let

$$\varphi(\theta) = \int_0^\infty e^{-\theta u}\, dF(u), \tag{4.3.13}$$

and θ_0 be the non-zero solution (if it exists) of

$$\exp(\theta_0 b)\, \varphi(\theta_0) = 1. \tag{4.3.14}$$

We then have, for $y > 0$ and $K \geq 3$,

$$\Pr\{S_{M(y)} > (K-2)b-y'\} \doteq \begin{cases} y'/(K-2)b, & \mathcal{E}(u_1) = b, \\ \dfrac{1 - \exp(-\theta_0 y')}{\exp(\theta_0\,[(K-2)b-y']) - \exp(-\theta_0 y')}, & \mathcal{E}(u_1) \neq b, \end{cases} \tag{4.3.15}$$

and

$$\mathcal{E}(M(y)) \doteq y'[(K-2)b-y']/\mathrm{Var}(u_1), \qquad \mathcal{E}(u_1) = b,$$

$$\begin{aligned} \mathcal{E}(M(y)) \doteq\ & (1/(b-\mathcal{E}(u_1)))\{-y'(\exp(\theta_0[(K-2)b-y']) - 1) \\ & + [(K-2)b-y'](1 - \exp(-\theta_0 y'))\} \\ & \cdot \{\exp(\theta_0[(K-2)b-y']) - \exp(-\theta_0 y')\}^{-1}, \\ & \mathcal{E}(u_1) \neq b. \end{aligned} \tag{4.3.16}$$

By the use of Theorem 4.3.1, and equations (4.3.15) and (4.3.16), an approximate expression for $\mathcal{E}(N(y))$ may be obtained. However, if $y = 0$, then the above approximations give

$$\mathcal{E}(M(0)) = \Pr\{S_{M(0)} > (K-2)b\} \doteq 0.$$

Hence, the substitution of these quantities into the expression of Theorem 4.3.1 yields 0/0, an undefined quantity. To circumvent this difficulty, we write

$$\mathcal{E}(N(y)) = \mathcal{E}(M(y)) + \lim_{a \to 0} \frac{\mathcal{E}(M(a))\ \Pr\{S_{M(y)} \leq y'\}}{\Pr\{S_{M(a)} > (K-2)b-a'\}},$$

and substitute the approximations before taking the limit. We have after simplification, for $K \geq 3$,

$$\mathcal{E}(N(y)) \doteq [(K-2)b-y'][(K-2)b+y']/\mathrm{Var}(u_1), \qquad \mathcal{E}(u_1) = b,$$

$$\mathcal{E}(N(y)) \doteq [(K-2)b-y']/(b - \mathcal{E}(u_1))$$

$$- \frac{(\exp(\theta_0(K-2)b) - \exp(\theta_0 y'))}{\theta_0(b - \mathcal{E}(u_1))}, \qquad \mathcal{E}(u_1) \neq b. \qquad (4.3.17)$$

It will be shown in Theorem 4.3.3 that (4.3.17) can be used to put an upper bound on $\mathcal{E}(V_j)$. The next two lemmas and the theorem following them help us to attain this goal, while giving insight into why we have devoted much effort to obtain $\mathcal{E}(N(y))$.

Let

$$Y_j = W_{N_j}, \qquad j \geq 1, \qquad (4.3.18)$$

(so that Y_j is the amount of waiting time in the system immediately prior to the j-th balk).

Lemma 4.3.3

The stochastic process $\{Y_j;\ j = 1,2,3,\ldots\}$ is a time-homogeneous Markov process on the continuous state space $((K-2)b,\ (K-1)b]$.

Proof of Lemma 4.3.3

By the definition of N_j in Section 4.1, the state space is as described. For $1 \le i_1 < i_2 < \ldots < i_m < j$, by the Markov and time-homogeneity properties of W_n, we have

$$\begin{aligned}
&\Pr\{Y_{j+1} \le x \mid Y_j = y,\ Y_{i_m} = y_m, \ldots,\ Y_{i_1} = y_1\} \\
&\qquad = \Pr\{W_{N_{j+1}} \le x \mid W_{N_j} = y,\ W_{N_{i_m}} = y_m, \ldots,\ W_{N_{i_1}} = y_1\} \\
&\qquad = \Pr\{W_{N_2} \le x \mid W_{N_1} = y\} \\
&\qquad = \Pr\{Y_2 \le x \mid Y_1 = y\},
\end{aligned}$$

thus completing the proof.

Lemma 4.3.4

(a) The distribution of M_{j+1} conditional on $Y_j = y$ is the same as that of $N(y)$ for $j \ge 1$.

(b) The distribution of V_{j+1} conditional on $Y_j = y$ is the same as that of $\sigma_{N(y)} - \sigma_0$ for $j \ge 1$.

(c) $\min_{(K-2)b<y\le(K-1)b} \mathcal{E}(N(y)) \le \mathcal{E}(M_j) \le \max_{(K-2)b<y\le(K-1)b} \mathcal{E}(N(y)), \quad j \ge 2.$

Proof of Lemma 4.3.4

By the Markov nature of W_n, we have

$$\Pr\{M_{j+1} = n \mid Y_j = y\}$$

$$= \Pr\{W_{N_j+1} \le (K-2)b, \ldots, W_{N_j+n-1} \le (K-2)b, W_{N_j+n} > (K-2)b \mid W_{N_j} = y\}$$

$$= \Pr\{W_1 \le (K-2)b, \ldots, W_{n-1} \le (K-2)b, W_n > (K-2)b \mid W_0 = y\}$$

$$= \Pr\{N(y) = n\},$$

completing part (a) of the lemma.

Further, since $u_1, u_2, u_3, \ldots$, are mutually independent and identically distributed,

$$\Pr\{V_{j+1} \mid Y_j = y\} = \Pr\{u_{N_j+1} + \ldots + u_{N_{j+1}} \le x \mid W_{N_j} = y\}$$

$$= \sum_{n=1}^{\infty} \Pr\{u_{N_j+1} + \ldots + u_{N_j+n} \le x, M_{j+1} = n \mid W_{N_j} = y\}$$

$$= \sum_{n=1}^{\infty} \Pr\{\sigma_n - \sigma_0 \le x, N(y) = n\}$$

$$= \Pr\{\sigma_{N(y)} - \sigma_0 \le x\},$$

and the (b) part is proved.

Finally, from the (a) part of the lemma, we have, for $j \ge 1$,

$$\mathcal{E}(M_{j+1}) = \mathcal{E}_{Y_j}\{\mathcal{E}(N(Y_j))\}.$$

But, by Lemma 4.3.3, $(K-2)b < Y_j \le (K-1)b$, so that part (c) of the lemma follows. The proof is now complete.

Note that $Y_1, Y_2, Y_3, \ldots$, are not identically distributed unless Y_1 has the same distribution as the stationary distribution of Y_j.

Therefore, Lemma 4.3.4 implies that neither $M_2, M_3, M_4, \ldots$, nor $V_2, V_3, V_4, \ldots$, are identically distributed sequences of random variables in general.

Theorem 4.3.2

(a) $\mathcal{E}(\sigma_{N(y)} - \sigma_0) = \mathcal{E}(N(y))\ \mathcal{E}(u_1)$,

(b) $\mathcal{E}(V_j) = \mathcal{E}(M_j)\ \mathcal{E}(u_1)$, $\qquad j \geq 1$,

with u_1 given by (4.1.2).

Proof of Theorem 4.3.2

Let $\sigma_n^* = \sigma_n - \sigma_0$ so that

$$\sigma_n^* = u_1 + \ldots + u_n, \qquad n \geq 1.$$

The sequence $\sigma_n^* - n\ \mathcal{E}(u_1)$ forms a martingale and $\mathcal{E}(\sigma_n^* - n\ \mathcal{E}(u_1)) = 0$ for $n \geq 1$. Let $\underline{B}_k$ be the σ-field of events generated by $(\sigma_1^*, \ldots, \sigma_k^*)$, then the event $\{N(y) > k\} \in \underline{B}_k$ and $\underline{B}_k \subset \underline{B}_{k+1}$. Hence, $N(y)$ is an optional stopping rule and therefore has no effect on the martingale property. See, for example, Feller (1966). We now have $\mathcal{E}(\sigma_{N(y)}^* - N(y)\ \mathcal{E}(u_1)) = 0$, which establishes part (a) of the theorem.

By Lemma 4.3.4 and part (a) above, we can write

$$\begin{aligned}
\mathcal{E}(V_{j+1}) &= \underset{Y_j}{\mathcal{E}} \{\mathcal{E}(V_{j+1} \mid Y_j)\} \\
&= \underset{Y_j}{\mathcal{E}} \{\mathcal{E}(\sigma_{N(Y_j)} - \sigma_0)\} \\
&= \underset{Y_j}{\mathcal{E}} \{\mathcal{E}(N(Y_j))\ \mathcal{E}(u_1)\} \\
&= \mathcal{E}(u_1)\ \underset{Y_j}{\mathcal{E}} \{\mathcal{E}(N(Y_j))\} \\
&= \mathcal{E}(u_1)\ \mathcal{E}(M_{j+1}), \qquad j \geq 1.
\end{aligned}$$

Similarly, we have

$$\mathcal{E}(V_1) = \mathop{\mathcal{E}}_{W_0} (\sigma_{N(W_0)} - \sigma_0) = \mathcal{E}(M_1)\, \mathcal{E}(u_1).$$

The proof is now complete.

If one could obtain the exact value of $\mathcal{E}(M_j)$, then the above theorem implies $\mathcal{E}(V_j)$ could be easily found. However, the best we are able to do is obtain an approximate upper bound for $\mathcal{E}(V_j)$ when $K \geq 3$. An exact upper (and lower) bound for $\mathcal{E}(V_j)$ when $K = 2$ can be obtained directly from Theorem 4.3.1. No approximate lower bound for $\mathcal{E}(V_j)$ in the case $K \geq 3$ can be found since, when $y = (K-1)b$ (or equivalently $y' = (K-2)b$), the approximation (4.3.17) yields a value of zero. We now state formally the results that can be obtained. Since they hold for all $j \geq 2$, the steady state solution satisfies these bounds also.

Theorem 4.3.3

Let θ_0 satisfy (4.3.14).

If $K = 2$, then

$$\mathcal{E}(u_1)/F(b-0) \leq \mathcal{E}(V_j) \leq \mathcal{E}(u_1)\,[1 + 1/F(b-0)], \qquad j \geq 2.$$

If $K \geq 3$, then, approximately,

$$\mathcal{E}(V_j) \leq \mathcal{E}(u_1)\,(2k-5)\, b^2/\mathrm{Var}(u_1), \qquad \mathcal{E}(u_1) = b,$$

and

$$\mathcal{E}(V_j) \leq b\, \mathcal{E}(u_1)/(b-\mathcal{E}(u_1)) - \frac{\mathcal{E}(u_1)\,(\exp(\theta_0(K-2)b) - \exp(\theta_0(K-3)b)}{\theta_0(b - \mathcal{E}(u_1))},$$

$$\mathcal{E}(u_1) \neq b, \quad j \geq 2.$$

Proof of Theorem 4.3.3

By Lemma 4.3.4(c), we need the maximum and minimum of $\mathcal{E}(N(y))$ from Theorem 4.3.1 for y in the range $(K-2)b < y \leq (K-1)b$, or equivalently, $(K-3)b < y' \leq (K-2)b$ to obtain bounds for $\mathcal{E}(M_j)$. The exact bounds for $\mathcal{E}(N(y))$ when $K = 2$ are taken directly from the expression in the theorem, while the approximate results for $K \geq 3$ are obtained from (4.3.17). It is easily shown that the lower bounds are reached when $y' = (K-2)b$ and the upper bounds when $y' = (K-3)b$. By Theorem 4.3.2(b) we need only multiply these bounds by $\mathcal{E}(u_1)$ to complete the proof.

Finally, let $F_Y(\cdot)$ denote the stationary distribution of Y_j. That is, $F_Y(\cdot)$ satisfies

$$F_Y(x) = \int_{(K-2)b}^{(K-1)b} \Pr\{Y_2 \leq x \mid Y_1 = y\}\, dF_Y(y), \qquad (4.3.19)$$

when $(K-2)b < x \leq (K-1)b$. If Y_1 has the distribution $F_Y(\cdot)$, then it is well known that Y_j has the distribution $F_Y(\cdot)$ for all j. See, for example, Feller (1966). We then have, by Lemma 4.3.4, the following theorem.

Theorem 4.3.4

If Y_1 has the distribution $F_Y(\cdot)$ that satisfies (4.3.19), then

(a) $M_2, M_3, M_4, \ldots$, are identically distributed,

(b) $V_2, V_3, V_4, \ldots$, are identically distributed, and

(c) $$\mathcal{E}(M_j) = \int_{(K-2)b}^{(K-1)b} \mathcal{E}(N(y))\, dF_Y(y), \qquad j \geq 2,$$

with $\mathcal{E}(N(y))$ given by Theorem 4.3.1.

CHAPTER 5

THE INVENTORY PROBLEM: DISCRETE CASE

5.1 Definition of the Inventory System

We suppose there exists a subwarehouse, maintaining an inventory of finite capacity S, that holds material (discrete) for future demand. We assume the item-by-item demand for the stored objects occurs according to the stochastic process $\{D(t);\ t \geq 0\}$ defined by

$$D(t) = \sum_{j=1}^{\infty} U(t - \tau_j) \tag{5.1.1}$$

with $U(\cdot)$ the unit step function at zero. It will be assumed that the inter-demand times, $\tau_2 - \tau_1$, $\tau_3 - \tau_2$, $\tau_4 - \tau_3, \ldots$, for the items in storage are mutually independent and that the distribution of $\tau_j - \tau_{j-1}$ is given by

$$\Pr\{\tau_j - \tau_{j-1} \leq u\} = G(u), \qquad u \geq 0, \quad j \geq 2. \tag{5.1.2}$$

In order to maintain a stock on hand, the subwarehouse places an order for replacement items to a warehouse. It will be held that items are so ordered in lots of integral size $\nu (1 \leq \nu \leq S)$ and that orders are placed at the times $\sigma_1, \sigma_2, \sigma_3, \ldots$, with σ_j defined by

$$\sigma_j = \inf \{t \mid D(t) = j\nu\}, \qquad j \geq 1. \tag{5.1.3}$$

From the definition of D(t), we have $\sigma_j = \tau_{j\nu}$ so that $\sigma_2 - \sigma_1$, $\sigma_3 - \sigma_2$, $\sigma_4 - \sigma_3, \ldots$, are mutually independent and $\sigma_j - \sigma_{j-1}$ has distribution

$$\Pr\{\sigma_j - \sigma_{j-1} \leq x\} = G_\nu(x), \qquad x \geq 0, \quad j \geq 2, \tag{5.1.4}$$

where $G_\nu(\cdot)$ is the ν-th convolution of $G(\cdot)$ with itself.

Let $\{S(t);\ t \geq 0\}$ be the stochastic process such that $S(t)$ represents the inventory or stock level in the subwarehouse at time t. If we let $\{R(t);\ t \geq 0\}$ be the stochastic process such that $R(t)$ is the number of orders filled by the warehouse in $[0,t]$ for our subwarehouse of interest, then $S(t)$ will be defined by

$$S(t) = S - D(t) + \nu R(t). \tag{5.1.5}$$

The above definition assumes that the inventory is initially full, i.e., $S(0) = S$.

An order for replacement stock of lot size ν, made at time σ, may be one of two types. We have a "regular" order provided that $S(\sigma) > S - \nu[S/\nu]$, where $[x]$ means the integral part of x. In this case, the time to fill an order (hereafter, the service time) is assumed to be a random variable. The successive regular service times, denoted by $w_1, w_2, w_3, \ldots$, are assumed to be mutually independent and independent of the demand process $D(t)$. The distribution function of w_j is given by

$$\Pr\{w_j \leq w\} = H(w), \qquad w \geq 0, \quad j \geq 1. \tag{5.1.6}$$

We have an "emergency" order if $S(\sigma) = S - \nu[S/\nu]$. In this case, the emergency service time is supposed instantaneous, or at least effectively zero, so that $S(\sigma + 0) = S - \nu[S/\nu] + \nu$.

In other words, regular ordering procedures are used provided that at the time we place such an order, there are at least ν items in the subwarehouse. If there are less than ν items in the subwarehouse

when an order is placed, we utilize emergency measures to obtain the lot of ν items. Utilizing this ordering scheme, we avoid the disaster of running completely out of stock in the subwarehouse. Figure 5.1 gives a typical realization of $S(t)$.

The behavior of the warehouse in filling the regular orders is important to a discussion of the inventory problem. It will be assumed that the warehouse operates under one of two distinct systems. Under the first system, the warehouse can handle only one order at a time, so that successive orders, which arrive while an order is being filled, form a queue and must wait to begin being processed or "served." The orders are then processed by the warehouse according to a strict rotation basis of "first come, first served." The warehouse just described will be called the one-server warehouse. Under the second system, an order begins processing as soon as it arrives in the warehouse so that no order must wait for "service." A warehouse operating under this procedure will be called an infinite-server warehouse. We shall consider both one-server and infinite-server warehouses.

We now state the following formal definition of the concepts discussed so far.

Definition 5.1.1

The ordering scheme $(G,H,S,\nu,1)$ is a policy for maintaining the level of inventory in a subwarehouse where:

(a) The capacity of the inventory is S.

(b) Item-by-item demand for objects in storage satisfies (5.1.1) and the inter-demand times are mutually independent random variables with distribution function $G(\cdot)$.

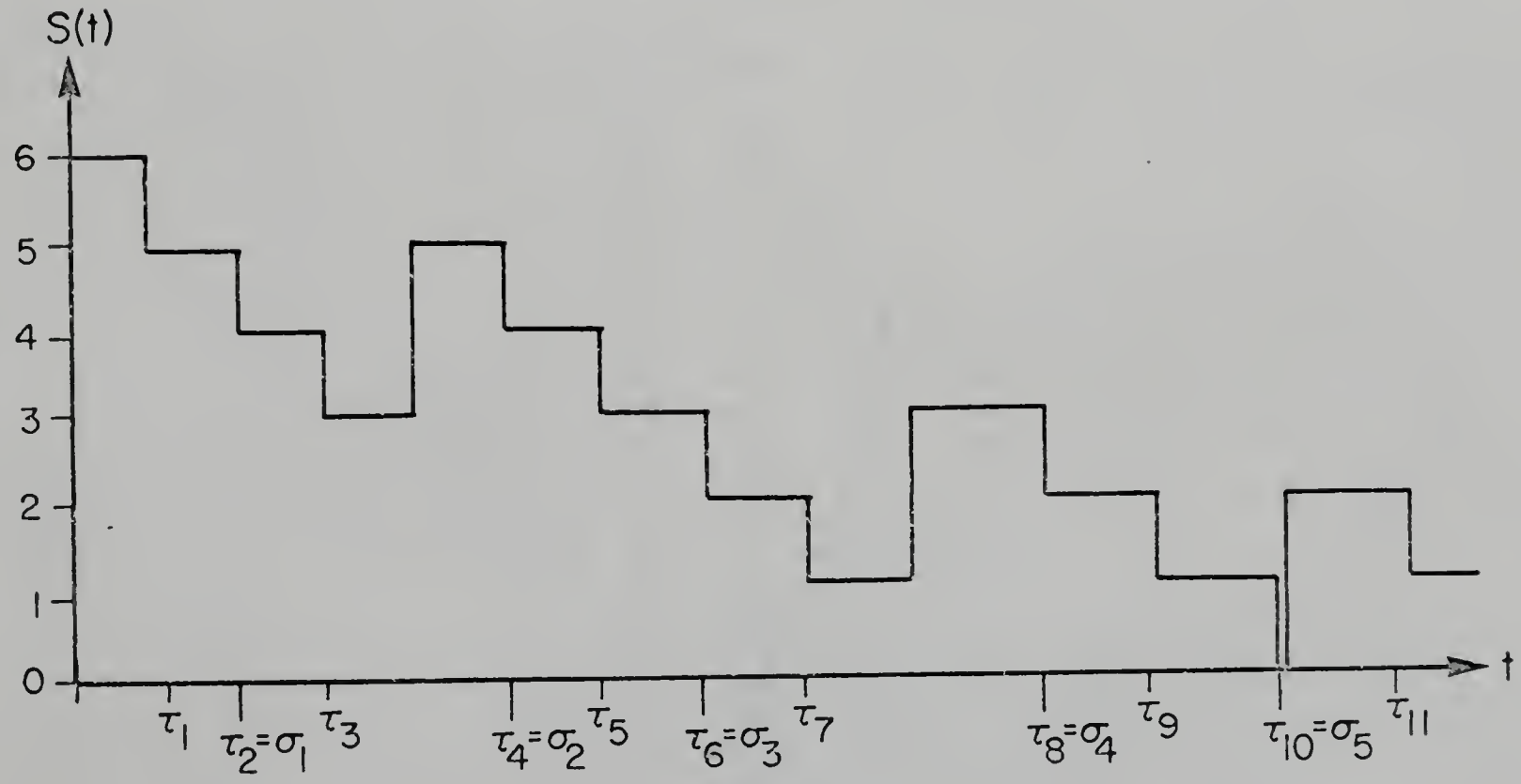

Figure 5.1. A Typical Path of S(t) when S = 6 and ν = 2.

(c) Lots of ν items ($1 \leq \nu \leq S$) are ordered at the times given by (5.1.3).

(d) The orders are made to a one-server warehouse.

(e) Regular service times are mutually independent random variables, are independent of the demand process, and possess a distribution function $H(\cdot)$.

(f) Instantaneous service occurs for orders placed when less than ν items remain in storage at the time an order is placed.

If we change condition (d) of the definition to state that orders are made to an infinite-server warehouse, we have the ordering scheme (G,H,S,ν,∞).

Clearly, the cost of maintaining the inventory level in the subwarehouse will be a function of ν, the lot size ordered. The optimal value of the lot size is defined herein to be that value of ν which minimizes the cost. It has to be remembered, however, that frequently not all values of ν are available to us since orders to the warehouse may have to be in multiples of ten, a dozen, a gross, or some other basic unit. Our best ν is that of finding the optimal attainable value of ν. In Section 5.3, a cost function is defined that utilizes reasonable costs associated with maintaining the inventory level.

While searching for a minimum cost with respect to ν, ν may take all values from 1 to S. Therefore, the distribution on regular service times could quite possibly be a function of ν, the lot size ordered.

5.2 Relation of the Inventory System to Queues with Balking

Recall from Section 5.1 that $S(\sigma_k)$ is the stock level at the time the k-th order is placed. $S(\sigma_k)$ tells us whether an order is regular or emergency. Since, realistically, emergency orders have large costs (more than the costs of regular orders), the value of $S(\sigma_k)$ is of extreme importance in determining the cost of maintaining the inventory level in the subwarehouse. A study of the properties of $S(\sigma_k)$ can be facilitated by making the following observations. From equations (5.1.5) and (5.1.3), we have

$$\begin{aligned} S(\sigma_k) &= S - D(\sigma_k) + \nu R(\sigma_k) \\ &= S - \nu\{k - R(\sigma_k)\}. \end{aligned} \tag{5.2.1}$$

Define the stochastic process $\{Q(t);\ t \geq 0\}$ by

$$Q(t) = [D(t)/\nu] - R(t), \tag{5.2.2}$$

where $[x]$ is the integral part of x, so that $Q(t)$ represents the number of unfilled orders, for our subwarehouse of interest, at time t. From (5.2.1) and (5.2.2), we have

$$S(\sigma_k) = S - \nu Q(\sigma_k) \tag{5.2.3}$$

so that a knowledge of $Q(\sigma_k)$ gives us the value of $S(\sigma_k)$. Therefore, a study of the stochastic process $\{Q(t);\ t \geq 0\}$ is needed. It will be demonstrated in Theorem 5.2.1 that such a study has been carried out for some special cases in Chapters 2 through 4.

For the cost function to be defined in Section 5.3, we will make use of the following random variables. Define

$$N_1 = \inf \{k > 0 | S(\sigma_k) = S - \nu[S/\nu]\},$$

$$N_n = \inf \{k > N_{n-1} | S(\sigma_k) = S - \nu [S/\nu]\}, \quad n \geq 2 \qquad (5.2.4)$$

(so that N_n is the number of orders, regular and emergency, placed up to and including the n-th emergency order), and

$$V_1 = \sigma_{N_1},$$

$$V_j = \sigma_{N_j} - \sigma_{N_{j-1}}, \qquad j \geq 2 \qquad (5.2.5)$$

(so that V_1 is the time until the first emergency order is placed and V_j $(j \geq 2)$ is the time between the (j-1)-st and j-th emergency orders).

Theorem 5.2.1

For an ordering scheme $(G,H,S,\nu,1)$ ((G,H,S,ν,∞)) we have the following dualities.

(a) $Q(t)$ is the number of people in the system at time t for the queue $G_\nu/H/1$ $(G_\nu/H/\infty)$ with balking at queues of length $[S/\nu] - 1$.

(b) N_k is the number of people who arrive in the system up to and including the k-th person to balk in the queue $G_\nu/H/1$ $(G_\nu/H/\infty)$ with balking at queues of length $[S/\nu] - 1$.

(c) V_1 is the time until the first balk and V_j $(j \geq 2)$ is the time between the (j-1)-st and j-th balks in the queue $G_\nu/H/1$ $(G_\nu/H/\infty)$ with balking at queues of length $[S/\nu] - 1$.

Proof of Theorem 5.2.1, Part (a)

By definition (5.2.2), we have

$$Q(t) = [D(t)/\nu] - R(t).$$

Now $[D(t)/\nu]$ has unit increases at the times $\sigma_1, \sigma_2, \sigma_3, \ldots$, so that $Q(t)$ also has unit increases at these times. Hence, the order time σ_j can be considered as the arrival time of a "customer" into the warehouse.

$R(t)$, by definition, is the number of orders filled by the warehouse in $[0,t]$. The time it takes to fill a regular order is w_j. Since $R(t)$ increases by a unit amount at the time an order is filled, $Q(t)$ decreases by a unit amount at that time. Therefore, w_j is the service time of a "customer" in the warehouse.

Finally, by the restriction of emergency orders and (5.1.5), σ_k is such that

$$S(\sigma_k + 0) = S - \nu[S/\nu] + \nu \quad \text{when} \quad S(\sigma_k) = S - \nu\,[S/\nu],$$

if and only if

$$R(\sigma_k + 0) = R(\sigma_k) + 1.$$

Hence, from (5.2.2) and (5.2.3)

$$S(\sigma_k + 0) = S - \nu[S/\nu] + \nu \quad \text{when} \quad S(\sigma_k) = S - \nu[S/\nu]$$

if and only if

$$Q(\sigma_k + 0) = [S/\nu] - 1 \quad \text{when} \quad Q(\sigma_k) = [S/\nu].$$

Therefore, a "customer" balks at the queue of length $[S/\nu] - 1$.

By the assumptions placed on the ordering times and the service times for the ordering scheme, and the discussion of a queue with balking in Section 2.1, we complete part (a) of the proof.

Proof of Theorem 5.2.1, Parts (b) and (c)

Simply note by (5.2.3) that

$$N_1 = \inf \{k > 0 | Q(\sigma_k) = [S/\nu]\},$$

$$N_n = \inf \{k > N_{n-1} | Q(\sigma_k) = [S/\nu]\}, \qquad n \geq 2.$$

By part (a) of the theorem and definitions in Section 2.1, we complete the proof.

5.3 The Cost Function $C(\nu)$

For the inventory problem discussed in Section 5.1, consider the following costs associated with running the ordering scheme $(G,H,S,\nu,1)$ or (G,H,S,ν,∞):

C_0: The cost of placing an order,

C_1: The per unit cost of the commodity, and

C_2: A penalty cost for instantaneous delivery of an emergency order that is possibly a function of ν, the lot size ordered.

Define the stochastic processes $\{N(t); t \geq 0\}$ and $\{M(t); t \geq 0\}$ by

$$N(t) = \sum_{j=1}^{\infty} U(t - \sigma_j) \tag{5.3.1}$$

and

$$M(t) = \sum_{j=1}^{\infty} U(t - \sigma_{N_j}) \tag{5.3.2}$$

with σ_j given by (5.1.3) and N_j by (5.2.4). Then $N(t)$ is the total number of orders placed in the interval $[0,t]$ (regular and emergency)

and M(t) is the number of these that are emergency orders. Definitions (5.3.1) and (5.3.2) are not the same random variables as defined by (4.3.1) and (4.3.2), respectively.

Let

$$C(\nu;t) = (C_0 + C_1\nu)\, N(t) + C_2 M(t) \tag{5.3.3}$$

so that $C(\nu;t)$ is the total cost of ordering lots of size ν during the time interval $[0,t]$. Since $N(t)$ and $M(t)$ are random quantities, we shall concern ourselves with the expected total cost, $\mathcal{E}\{C(\nu;t)\}$, during the interval $[0,t]$. Further, the ν which minimizes $\mathcal{E}\{C(\nu;t_0)\}$ for a fixed t_0 will minimize $\mathcal{E}\{C(\nu;t_0)\}/t_0$, so that we shall restrict ourselves to the latter quantity. Finally, since a subwarehouse that maintains an inventory is usually established with the thought of operating for a long period of time, we choose to minimize the expected total cost of ordering per unit time in the long run, a quantity that is mathematically tractable. That is, we want the value of ν ($\nu = 1,2,\ldots,$ or S) that minimizes

$$\begin{aligned} C(\nu) &= \lim_{t\to\infty} \mathcal{E}\{C(\nu;t)/t\} \\ &= (C_0 + C_1\nu) \lim_{t\to\infty} \mathcal{E}\{N(t)/t\} \\ &\qquad + C_2 \lim_{t\to\infty} \mathcal{E}\{M(t)/t\}. \end{aligned} \tag{5.3.4}$$

But $\sigma_2 - \sigma_1$, $\sigma_3 - \sigma_2$, $\sigma_4 - \sigma_3, \ldots$, are mutually independent and identically distributed random variables. Therefore, N(t) is a (delayed) renewal process and, by the Elementary Renewal Theorem, Prabhu (1965a), we have

Lemma 5.3.1

$$\lim_{t\to\infty} \mathcal{E}\{N(t)/t\} = 1/\mathcal{E}(\sigma_2 - \sigma_1)$$

$$= 1/\nu\xi \tag{5.3.5}$$

with

$$\xi = \int_0^\infty u \, dG(u). \tag{5.3.6}$$

A similar closed form for $\lim_{t\to\infty} \mathcal{E}\{M(t)/t\}$ does not exist for an arbitrary ordering scheme $(G,H,S,\nu,1)$ or (G,H,S,ν,∞). The reason for this is that $M(t)$ is a function of the random variables N_j whose properties depend heavily on the distribution of service times and whether we have a one-server or infinite-server warehouse.

In the following sections, we consider reasonable candidates for the distribution function, $H(\cdot)$, on the service times and both one-server and infinite-server warehouses. For the cases discussed in these sections, a "closed" form for $\lim_{t\to\infty} \mathcal{E}\{M(t)/t\}$ will be obtained.

At this juncture, it should be pointed out that when $[S/\nu] = 1$, Theorem 5.2.1 gives $N_k = k$. Therefore, $M(t) = N(t)$ and $\lim_{t\to\infty} \mathcal{E}\{M(t)/t\} = 1/\nu\xi$. For the future we shall therefore concern ourselves with the cases $[S/\nu] \geq 2$.

5.4 Solution of C(ν) Using the Queue GI/M/1 with Balking

General Demand Function

In this section, we develop the solution of the cost function C(ν) for the ordering scheme (G,M,S,ν,1).

The subwarehouse places orders of lot size ν with a one-server warehouse, so that orders arrive at the warehouse, form a queue, and are processed on a strict "first come, first served" basis. We are leaving the item-by-item demand function D(t) general, but we are requiring that regular orders have service times with a Markov, or negative exponential, distribution. Therefore, the distribution of w_j is

$$\Pr\{w_j \le w\} = H(w) = 1 - e^{-\lambda w}, \qquad w \ge 0, \quad j \ge 1, \tag{5.4.1}$$

where $1/\lambda$ is the mean service time to process an order.

It is reasonable that the time to fill an order, w_j, should depend in some manner on ν, the lot size of the order placed. We may allow for this by permitting that λ be a function of ν. Typically, we may have $\lambda = \alpha/\nu$, α a constant. In order to find the value of ν which minimizes C(ν) of Section 5.3, we prove the following theorem.

Theorem 5.4.1

For the ordering scheme (G,M,S,ν,1), the cost function C(ν) of Section 5.3 has the form

$$C(\nu) = (C_0 + \nu C_1)/\xi\nu + C_2/\xi\nu\ \mu(\nu) \tag{5.4.2}$$

with

$$\mu(\nu) = \begin{cases} C_Z^{([S/\nu]-2)} \; 1/\{[\psi(\lambda(1-Z))]^{\nu} - Z\}, & [S/\nu] \geq 2, \\ 1, & [S/\nu] = 1, \end{cases}$$

where

$$\psi(\theta) = \int_0^{\infty} e^{-\theta u} \, dG(u),$$

and

$$\xi = \int_0^{\infty} u \, dG(u).$$

Proof of Theorem 5.4.1

By Lemma 5.3.1, we need only find $\lim_{t \to \infty} \mathcal{E}\{M(t)/t\}$ to complete the proof. Now

$$M(t) = \sum_{j=1}^{\infty} U(t - \sigma_{N_j}).$$

Let $V_1 = \sigma_{N_1}$ and $V_j = \sigma_{N_j} - \sigma_{N_{j-1}}$ $(j \geq 2)$. By Theorem 5.2.1(c), σ_{N_j} is the time until the j-th balk occurs in the queue $G_\nu/M/1$ with balking at queues of length $[S/\nu] - 1$. Hence, we can apply the results of Chapter 2 with

$$K = [S/\nu],$$

$$F(x) = G_\nu(x), \quad \text{and}$$

$$\varphi(\theta) = [\psi(\theta)]^{\nu}.$$

From Theorem 2.3.2, we therefore have M(t) is a delayed renewal process. By the Elementary Renewal Theorem, Prabhu (1965a), we have

$$\lim_{t\to\infty} \mathcal{E}\{M(t)/t\} = 1/\mathcal{E}(V_2)$$

$$= 1/\nu \; \xi \; \mu(\nu)$$

with $\mu(\nu)$ from Theorem 2.2.2. The proof is now complete.

Writing (5.4.2) out, we see that $C(\nu)$ is minimized when

$$h(\nu) = (1/\nu)\{1 + (C_2/C_0)/\mu(\nu)\} \tag{5.4.3}$$

is minimized. Therefore, $h(\nu)$ can be considered as the cost function of interest.

Example: Poisson Demand

Assume the item-by-item demand for objects in the subwarehouse occurs according to a Poisson process with intensity μ. Then

$$\Pr\{D(t) = n\} = e^{-\mu t} (\mu t)^n/n! , \qquad n \geq 0,$$

and the inter-demand times have the distribution

$$\Pr\{\tau_j - \tau_{j-1} \leq x\} = G(x) = 1 - e^{-\mu x}, \quad x \geq 0, \quad j \geq 2. \tag{5.4.4}$$

That is, we have the ordering scheme (M,M,S,ν,1).

To apply Theorem 5.4.1, we need a workable expression for $\mu(\nu)$.

Define

$$\rho = \mu/\lambda$$

and

$$p = \rho/(1 + \rho), \quad q = 1/(1 + \rho).$$

From (5.4.4), we have

$$\psi(\theta) = (1 + \theta/\mu)^{-1} .$$

Hence,

$$[\psi(\lambda(1 - Z))]^{\nu} = (1 + (1 - Z)\lambda/\mu)^{-\nu}$$

$$= (1 + (1 - Z)/\rho)^{-\nu}$$

$$= p^{\nu}/(1 - Zq)^{\nu}.$$

Therefore,

$$\mu(\nu) = C_Z^{([S/\nu]-2)} \frac{1}{p^{\nu}/(1 - qZ)^{\nu} - Z}$$

$$= C_Z^{([S/\nu]-2)} \left(\frac{1 - qZ}{p}\right)^{\nu} \Big/ \left\{1 - Z\left(\frac{1 - qZ}{p}\right)^{\nu}\right\}$$

$$= C_Z^{([S/\nu]-2)} \sum_{j=0}^{\infty} Z^j \left(\frac{1 - qZ}{p}\right)^{(j+1)\nu}, \quad |Z| < p^{\nu}$$

$$= \sum_{j=0}^{[S/\nu]-2} C_Z^{([S/\nu]-j-2)} [(1 - qZ)/p]^{(j+1)\nu}$$

$$= \sum_{j=0}^{[S/\nu]-2} \binom{(j+1)\nu}{[S/\nu]-j-2} (-q)^{[S/\nu]-2-j} p^{-(j+1)\nu}.$$

As an illustration, Table 5.1 gives the values of $h(\nu)$ of equation (5.4.3) for $C_2/C_0 = 10.0$ and various values of ρ and S. It is to be noted that considerable savings can be effected by the proper choice of ν, the lot size ordered to replenish the stock.

TABLE 5.1

VALUES OF h(ν) FOR THE ORDERING SCHEME (M,M,S,ν,1)
$C_2/C_0 = 10.0$

S	8		9		10	
ρ / ν	0.8	1.2	0.8	1.2	0.8	1.2
1	1.5040	3.1717	1.3876	3.0673	1.3007	2.9877
2	0.5672	0.7759	0.5672	0.7759	0.5184	0.6313
3	0.6260	0.8743	0.3634	0.4461*	0.3634	0.4461
4	0.3475*	0.4713*	0.3475*	0.4713	0.3475	0.4713
5	2.2000	2.2000	2.2000	2.2000	0.2347*	0.2966*
6	1.8333	1.8333	1.8333	1.8333	1.8333	1.8333
7	1.5714	1.5714	1.5714	1.5714	1.5714	1.5714
8	1.3750	1.3750	1.3750	1.3750	1.3750	1.3750
9			1.2222	1.2222	1.2222	1.2222
10					1.1000	1.1000

*Denotes optimal value.

5.5 Solution of C(ν) Using the Queue GI/M/∞ with Balking

General Demand Function

In this section, we develop the solution of the cost function C(ν) for the ordering scheme (G,M,S,ν,∞).

The subwarehouse places orders of lot size ν with an infinite-server warehouse, so that when an order is received at the warehouse, processing begins immediately. Once again, we hold the item-by-item demand function D(t) general, but we require that regular orders have service times with the Markov, or negative exponential, distribution. Therefore, the distribution of w_j is

$$\Pr\{w_j \le w\} = H(w) = 1 - e^{-\lambda w}, \qquad w \ge 0, \quad j \ge 1, \tag{5.5.1}$$

where $1/\lambda$ is the mean service time to process an order.

As before, it is reasonable that the time to fill an order should depend in some manner on ν, the lot size ordered. We may allow for this by permitting λ to be a function of ν. Typically, we may have $\lambda = \alpha/\nu$, α a constant. In order to find the optimal value of ν, we prove the following theorem.

Theorem 5.5.1

For the ordering scheme (G,M,S,ν,∞), the cost function C(ν) of Section 5.3 has the form

$$C(\nu) = (C_0 + \nu C_1)/\nu \xi + C_2/\nu \xi \mu(\nu) \tag{5.5.2}$$

with

$$\mu(\nu) = \begin{cases} m([S/\nu],\nu), & [S/\nu] \ge 2, \\ 1, & [S/\nu] = 1, \end{cases}$$

where $m(2,\nu),\ldots,m([S/\nu],\nu)$ satisfy the relationships

$$m(2,\nu) = [\psi(\lambda)]^{-\nu},$$

$$m(n+1,\nu) = [\psi(n\lambda)]^{-\nu}\Big[1 + \sum_{k=2}^{n} b_{\nu}(n,n-k+2) \sum_{j=k}^{n} m(j,\nu)\Big],$$

$$n = 2,3,\ldots, [S/\nu] - 1,$$

with

$$\psi(\theta) = \int_0^{\infty} e^{-\theta u}\, dG(u),$$

$$\xi = \int_0^{\infty} u\, dG(u),$$

and

$$b_{\nu}(n,n-k+2) = \binom{n}{k-2} \sum_{j=0}^{n-k+2} \binom{n-k+2}{j}(-1)^j\, [\psi(\lambda(j+k-2))]^{\nu}.$$

Proof of Theorem 5.5.1

By Lemma 5.3.1, we need only find $\lim_{t\to\infty} \mathcal{E}\{M(t)/t\}$ to complete the proof.

Recall that

$$M(t) = \sum_{j=0}^{\infty} U(t - \sigma_{N_j}).$$

Let $V_1 = \sigma_1$ and $V_j = \sigma_{N_j} - \sigma_{N_{j-1}}$ $(j \geq 2)$. By Theorem 5.2.1(c), σ_{N_j} is the time until the j-th balk occurs in the queue $G_{\nu}/M/\infty$ with balking at queues of length $[S/\nu] - 1$. The results of Chapter 3 apply here, with

$$K = [S/\nu],$$

$$F(x) = G_{\nu}(x), \quad \text{and}$$

$$\varphi(\theta) = [\psi(\theta)]^{\nu}.$$

From Theorem 3.3.2, we therefore have M(t) is a delayed renewal process. By the Elementary Renewal Theorem, Prabhu (1965a),

$$\lim_{t\to\infty} \mathcal{E}\{M(t)/t\} = 1/\nu \; \xi \; \mu(\nu).$$

The quantities used in calculating $\mu(\nu)$ follow from equations (3.2.6) and (3.2.8). The proof is now complete.

Writing (5.5.2) out, we see that $C(\nu)$ is minimized when

$$h_1(\nu) = (1/\nu)\{1 + (C_2/C_0)/\mu(\nu)\} \qquad (5.5.3)$$

is minimized. Therefore, $h_1(\nu)$ can be considered as the cost function of interest.

Example: Poisson Demand

Assume the item-by-item demand for objects in the subwarehouse occurs according to a Poisson process with intensity μ. Then

$$\Pr\{D(t) = n\} = e^{-\mu t} (\mu t)^n/n! \;, \quad n \geq 0,$$

and the inter-demand times have the distribution

$$\Pr\{\tau_j - \tau_{j-1} \leq x\} = G(x) = 1 - e^{-\mu x}, \qquad x \geq 0, \quad j \geq 2. \quad (5.5.4)$$

That is, we have the ordering scheme (M,M,S,ν,∞).

Define $\rho = \mu/\lambda$. From (5.5.4), we have

$$\psi(\theta) = (1 + \theta/\mu)^{-1}.$$

Hence, $\mu(\nu)$ of Theorem 5.5.1 becomes a function of

$$\psi(\nu\lambda) = (1 + n\,\lambda/\mu)^{-1}(1 + n/\rho)^{-1},$$

which is a function only of ρ and not μ and λ separately.

As an illustration, Table 5.2 gives the values of $h_1(\nu)$ of equation (5.5.3) for $C_2/C_0 = 10.0$ and various values of ρ and S. Once again, considerable savings can be effected by the proper choice of ν, the lot size ordered.

Comparison of Sections 5.4 and 5.5

We note that Sections 5.4 and 5.5 both deal with inventories subject to general demand and negative expoenetial regular service times. Whereas the results of Section 5.4 are based on orders being placed to a one-server warehouse, Section 5.5 assumes we have an infinite-server warehouse so that processing of an order begins as soon as it is received.

These two sections represent the extreme cases in terms of the number of processors available in a warehouse to process an order when we have general item-by-item demand and negative exponential service times. Therefore, we can calculate the optimal value of ν, the lot size ordered, for the best possible situation (processing begins immediately when an order is placed, Section 5.5) and the worst possible situation (an order must wait in turn before processing begins, Section 5.4).

It is worthwhile to compare Tables 5.1 and 5.2 for the case of Poisson item-by-item demand.

TABLE 5.2

VALUES OF $h_1(\nu)$ FOR THE ORDERING SCHEME (M,M,S,ν,∞)
$C_2/C_0 = 10.0$

S	8		9		10	
ν \ ρ	0.8	1.2	0.8	1.2	0.8	1.2
1	1.0002	1.0021	1.0000	1.0003	1.0000	1.0000
2	0.5044	0.5234	0.5044	0.5234	0.5001	0.5014
3	0.6260	0.8743	0.3406*	0.3654*	0.3406	0.3654
4	0.3475*	0.4713*	0.3475	0.4713	0.3475	0.4713
5	2.2000	2.2000	2.2000	2.2000	0.2347*	0.2966*
6	1.8333	1.8333	1.8333	1.8333	1.8333	1.8333
7	1.5714	1.5714	1.5714	1.5714	1.5714	1.5714
8	1.3750	1.3750	1.3750	1.3750	1.3750	1.3750
9			1.2222	1.2222	1.2222	1.2222
10					1.1000	1.1000

*Denotes optimal value.

5.6 Solution of C(ν) Using the Queue G1/D/1 with Balking

In this section, we consider the solution of C(ν) for the ordering scheme (G,D,S,ν,1).

The subwarchouse places orders of lot size ν with a one-server warehouse so that orders arrive at the warehouse, form a queue, and are processed on a strict "first come, first served" basis. We allow the item-by-item demand function D(t) to be general, but we require that regular orders have a constant service time b. It may be that b is a function of ν, the lot size ordered. Typically, we may allow for this by permitting $b = \nu d$, d a constant.

By Lemma 5.3.1, we need only find $\lim_{t\to\infty} \mathcal{E}\{M(t)/t\}$ to complete the solution of the cost function C(ν) defined in Section 5.3.

Recall that

$$M(t) = \sum_{j=1}^{\infty} U(t - \sigma_{N_j}).$$

From Theorem 5.2.1(c), σ_{N_j} is the time until the j-th balk occurs in the queue $G_\nu/D/1$ with balking at queues of length $[S/\nu] - 1$. Hence, the results of Chapter 4 apply here with $K = [S/\nu]$ and the distribution on inter-arrival times $F(x) = G_\nu(x)$.

Except for the trivial case $[S/\nu] = 1$ when $M(t) = N(t)$, it was noted in Section 4.3 that the random variables

$$V_j = \sigma_{N_j} - \sigma_{N_{j-1}}, \qquad j \geq 2,$$

do not have the property of identical distribution as was the case when the service times had a negative exponential distribution. Therefore,

M(t) is not a renewal process and $\mathcal{E}\{M(t)/t\}$ does not possess a simple limit. Since we are interested in the long run behavior of the inventory, we shall be content to utilize the steady state properties of the system and to redefine M(t) so that a suitable limit for $\mathcal{E}\{M(t)/t\}$ can be obtained.

Recall the definition of σ_n from Section 5.1. It is clear that σ_n is a function of ν, the lot size ordered. Define

$$S_n(\nu) = nb - \sigma_n, \qquad n \geq 1,$$

and

$$M^*(y;\nu) = \inf \{n > 0 | S_n(\nu) \leq -y \text{ or } S_n(\nu) > [S/\nu] - 2 - y\},$$

$$[S/\nu] \geq 3.$$

Furthermore, let $\{Y_j(\nu);\ j = 1,2,3,\ldots\}$ be the Markov process defined by (4.3.18). Y_j is written as $Y_j(\nu)$ to emphasize the dependence of the process on ν, the lot size ordered, when the quantities K and F(x) of Chapter 4 are $[S/\nu]$ and $G_\nu(x)$, respectively.

Dencte by $F_Y(y,\nu)$ the stationary distribution of $Y_j(\nu)$ so that $F_Y(y,\nu)$ satisfies

$$F_Y(y,\nu) = \int_{([S/\nu]-2)b}^{([S/\nu]-1)b} \Pr\{Y_2(\nu) \leq y | Y_1(\nu) = x\}\, d_x F_Y(x,\nu),$$

$$([S/\nu] - 2)b < y \leq ([S/\nu] - 1)b.$$

Finally, let

$$\mu(\nu) = 1, \quad [S/\nu] = 1,$$

$$= 1 + \int [1 - G_\nu(y-0)]/G_\nu(b-0)\, dF_Y(y,\nu), \qquad [S/\nu] = 2,$$

$$= \int \mathcal{E}(M^*(y-b;\ \nu))\, dF_Y(y,\nu)$$

$$+ [\mathcal{E}(M^*(0;\ \nu))/\Pr\{S_{M^*(0;\ \nu)}(\nu) > ([S/\nu] - 2)b\}]$$

$$\cdot \int \Pr\{S_{M^*(y-b;\ \nu)}(\nu) \le -y + b\}\, dF_Y(y,\nu), \quad [S/\nu] \ge 3. \qquad (5.6.1)$$

If $Y_1(\nu)$ possesses the stationary distribution $F_Y(y,\nu)$, then by Theorem 4.3.4, $\mu(\nu)$ is one plus the expected number of orders of lot size ν that occur between any two emergency orders when the inventory system is in the steady state.

Let $V_1^*, V_2^*, V_3^*, \ldots$, be a sequence of mutually independent random variables such that the distribution of V_j^* is

$$\Pr\{V_j^* \le x\} = \int \Pr\{V_{j+1} \le x \mid Y_j(\nu) = y\}\, dF_Y(y,\nu), \qquad j \ge 1.$$

By Theorem 4.3.4, $V_1^*, V_2^*, V_3^*, \ldots$, are identically distributed and represent the time between successive emergency deliveries when the system is in the steady state. For the purposes of this section, we redefine $M(t)$ as

$$M(t) = \sum_{j=1}^{\infty} U(t - V_1^* - \ldots - V_j^*), \qquad (5.6.2)$$

so that $M(t)$ is the number of emergency deliveries during $[0,t]$ when the inventory system is in the steady state.

Theorem 5.6.1

For the ordering scheme $(G,D,S,\nu,1)$, the cost function $C(\nu)$ of Section 5.3 (with $M(t)$ redefined by (5.6.2)) has the form

$$C(\nu) = (C_0 + C_1)/\xi\ \nu + C_2/\nu\ \xi\ \mu(\nu) \tag{5.6.3}$$

with

$$\xi = \int_0^\infty u\ dG(u)$$

and $\mu(\nu)$ given by (5.6.1).

Proof of Theorem 5.6.1

By the Elementary Renewal Theorem, Prabhu (1965a),

$$\lim_{t\to\infty} \mathcal{E}\{M(t)/t\} = 1/\mathcal{E}(V_1^*)$$

$$= 1/\nu\ \xi\ \mu(\nu),$$

the last equality following from Theorem 4.3.2 and the definition of V_j^*. Applying Lemma 5.3.1, we complete the proof.

Writing (5.6.3) out, we see that $C(\nu)$ is minimized when

$$h_2(\nu) = (1/\nu)\{1 + (C_2/C_0)/\mu(\nu)\} \tag{5.6.4}$$

is minimized, $h_2(\nu)$ may therefore be taken as the cost function of interest.

In many cases it may be difficult to obtain $\mu(\nu)$ when $[S/\nu] \geq 3$. Theorem 4.3.3 then gives a bound that may be used to obtain an approximate lower bound for $C(\nu)$.

CHAPTER 6

THE INVENTORY PROBLEM: CONTINUOUS CASE

6.1 First Passage Times of Non-Negative, Continuous Stochastic Processes with Infinitely Divisible Distributions

In this chapter, we wish to consider the ordering scheme for a subwarehouse that maintains an inventory of fluid material. We assume the demand for the fluid in storage occurs continuously. It is reasonable to further assume that the demand during any interval of time is independent of the demand during any other nonoverlapping interval of time and that the probability law for the demand during any interval $[s, s+t]$ is functionally dependent only on the length, t, of the interval. Hence, if $\{D(t);\ t \geq 0\}$ is the stochastic process such that $D(t)$ represents the demand for the fluid in storage during the time interval $[0, t]$, then we are assuming that $\{D(t);\ t \geq 0\}$ is a non-negative, continuous stochastic process with stationary, independent, nonoverlapping increments. By Theorem 2 of Feller (1966, p. 294), this is equivalent to stating that $\{D(t);\ t \geq 0\}$ is a non-negative, continuous stochastic process whose distribution is infinitely divisible. The distribution function of $D(t)$ will be denoted by

$$\Pr\{D(t) \leq x\} = \int_0^x g(y,t)\, dy, \qquad x \geq 0, \quad t \geq 0, \tag{6.1.1}$$

with $g(\cdot,t)$ a density on $[0,\infty)$ for each $t \geq 0$.

In the next section, a complete description of the ordering scheme used to replenish the inventory will be given. To solve this inventory problem, we need the probability law for the stochastic process $\{T(u);\ u \geq 0\}$ defined by

$$T(u) = \inf \{t \mid D(t) \geq u\} \tag{6.1.2}$$

(so that $T(u)$ is the first passage time of $D(t)$ into the interval $[u,\infty)$). The rest of the current section will be devoted to properties of $T(u)$.

Theorem 6.1.1

Let $\{T(u);\ u \geq 0\}$ be the stochastic process defined by (6.1.2), then

(a) $T(u)$ has stationary, independent, nonoverlapping increments, and

(b) The distribution of $T(u)$ is

$$\Pr\{T(u) \leq t\} = \int_u^\infty g(y,t)\, dy, \qquad t \geq 0, \quad u \geq 0.$$

Proof of Theorem 6.1.1

Since $T(u) \leq t$ if and only if $D(t) \geq u$,

$$\begin{aligned} \Pr\{T(u) \leq t\} &= \Pr\{D(t) \geq u\} \\ &= \int_u^\infty g(y,t)\, dy, \end{aligned}$$

completing part (b) of the proof.

To prove part (a), it is sufficient to show

$$\Pr\{T(y) - T(w) \leq s \mid T(w) = r\} = \Pr\{T(y-w) \leq s\} \tag{6.1.3}$$

for all $s \geq 0$, $r \geq 0$, and $0 \leq w < y$.

First we shall calculate $\Pr\{T(y) \leq t | T(w) = r\}$ for $w < y$ and $r < t$. Now, $\Pr\{T(w) = r\} = 0$ so that $\Pr\{T(y) \leq t | T(w) = r\}$ involves conditioning on an event of probability zero. Hence, the quantity $\Pr\{T(y) \leq t, T(w) = r\}/\Pr\{T(w) = r\}$ is undefined and therefore can not be used to define $\Pr\{T(y) \leq t | T(w) = r\}$. Cramér and Leadbetter (1967, pp. 219-222) give two plausible definitions for $\Pr\{T(y) \leq t | T(w) = r\}$ which are known as the vertical-window (v.w.) and horizontal-window (h.w.) conditional probabilities. These conditional probabilities are defined by

$$\Pr\{T(y) \leq t | T(w) = r\}_{v.w.}$$

$$= \lim_{\delta \to 0} \Pr\{T(y) \leq t | r \leq T(w) \leq r + \delta\} \qquad (6.1.4)$$

and

$$\Pr\{T(y) \leq t | T(w) = r\}_{h.w.}$$

$$= \lim_{\delta \to 0} \Pr\{T(y) \leq t | T(\tau) = r, \text{ for some } \tau \in [w, w+\delta]\}, \qquad (6.1.5)$$

respectively. Both (6.1.4) and (6.1.5) define $\Pr\{T(y) \leq t | T(w) = r\}$ in terms of a limit of conditional probabilities which involve conditioning on events of non-zero probability. Equation (6.1.4) is the usual definition of $\Pr\{T(y) \leq t | T(w) = r\}$. However, in our particular case (6.1.4) leads to an undefined quantity. Therefore, we choose to use the horizontal-window definition given by (6.1.5). We have, for $0 < \delta < y-w$,

$$\Pr\{T(y) \le t \mid T(w) = r\}_{h.w.}$$

$$= \lim_{\delta \to 0} \frac{\Pr\{T(y) \le t,\ T(\tau) = r \text{ for some } \tau \in [w, w+\delta]\}}{\Pr\{T(\tau) = r \text{ for some } \tau \in [w, w+\delta]\}}$$

$$= \lim_{\delta \to 0} \frac{\Pr\{D(t) \ge y,\ w \le D(r) \le w + \delta\}}{\Pr\{w \le D(r) \le w + \delta\}}$$

$$= \lim_{\delta \to 0} \frac{\int_{y}^{\infty} \int_{w}^{w+\delta} g(z-x, t-r)\, g(x,r)\, dx\, dz}{\int_{w}^{w+\delta} g(x,r)\, dx}$$

$$= \int_{y}^{\infty} g(z-w, t-r)\, dz[g(w,r)/g(w,r)]$$

$$= \int_{y-w}^{\infty} g(z, t-r)\, dz$$

$$= \Pr\{D(t-r) \ge y-w\}$$

$$= \Pr\{T(y-w) \le t-r\}. \qquad (6.1.6)$$

Let $t = r+s$ in equation (6.1.6), then

$$\Pr\{T(y) - T(w) \le s \mid T(w) = r\}_{h.w.}$$

$$= \Pr\{T(y) \le s+r \mid T(w) = r\}$$

$$= \Pr\{T(y-w) \le s\},$$

thus completing the proof.

The above theorem implies that $T(u)$ is also non-negative with an infinitely divisible distribution. Hence, the Laplace transforms of $T(u)$ and $D(t)$ have the forms

$$\mathcal{E}(e^{-\theta T(u)}) = e^{-w(\theta)u}, \qquad \theta > 0, \tag{6.1.7}$$

and

$$\mathcal{E}(e^{-\theta D(t)}) = e^{-v(\theta)t}, \qquad \theta > 0, \tag{6.1.8}$$

respectively, such that $w(\theta)$ and $v(\theta)$ are positive for $\theta > 0$ and possess completely monotone derivatives. See, for example, Feller (1966).

An attempt was made to find the relation between the Laplace transforms of (6.1.7) and (6.1.8) by utilizing the following technique. We have

$$\int_0^\infty \int_0^\infty e^{-\xi t}\, e^{-\theta y}\, \Pr\{T(y) \le t\}\, dt\, dy$$

$$= \int_0^\infty e^{-\theta y}\, e^{-w(\xi)y/\xi}\, dy$$

$$= 1/[\xi(\theta + w(\xi))], \tag{6.1.9}$$

and

$$\int_0^\infty \int_0^\infty e^{-\xi t}\, e^{-\theta y}\, \Pr\{D(t) \ge y\}\, dy\, dt$$

$$= \int_0^\infty e^{-\xi t}\, (1 - e^{-v(\theta)t})/\theta\, dt$$

$$= 1/\xi\theta - 1/[\theta(\xi + v(\theta))]$$

$$= v(\theta)/[\theta\xi(\xi + v(\theta))]. \tag{6.1.10}$$

Now $\Pr\{D(t) \ge y\} = \Pr\{T(y) \le t\}$ so that <u>if</u>

$$\int_0^\infty \int_0^\infty e^{-\xi t}\, e^{-\theta y}\, \Pr\{D(t) \ge y\}\, dy\, dt$$

$$= \int_0^\infty \int_0^\infty e^{-\xi t}\, e^{-\theta y}\, \Pr\{D(t) \ge y\}\, dt\, dy \tag{6.1.11}$$

(i.e., if we can change the order of integration), then (6.1.9) and (6.1.10) must be identical. Hence, we have

$$1/[\xi(\theta + w(\xi))] = v(\theta)/[\xi\ \theta(\xi + v(\theta))]$$

or

$$v(\theta)\ w(\xi) = \theta\ \xi$$

which implies

$$w(\xi) = c\ \xi$$

and

$$v(\theta) = \theta/c$$

for some $c > 0$. We must have by the uniqueness of Laplace transforms that

$$\Pr\{D(t) = t/c\} = \Pr\{T(y) = cy\} = 1.$$

It now seems that the only non-negative, continuous process $D(t)$ with stationary, independent, nonoverlapping increments is the trivial deterministic model $D(t) = t/c$. However, we know that if $D(t)$ has the gamma density

$$g(y,t) = e^{-y/\rho}\ y^{t-1}/\Gamma(t)\rho^{t}, \qquad y \geq 0, \tag{6.1.12}$$

then $D(t)$ is non-negative; continuous; has stationary, independent, nonoverlapping increments; and is clearly not deterministic. The point we make is that (6.1.11) is true only for the trivial case $D(t) = t/c$ and, hence, the order of integration can not be changed for any other choice of $D(t)$. So far we have been unable to find a suitable method for obtaining the Laplace transform of $T(u)$.

Since we would like as much information about T(u) as possible, it seems reasonable to try to calculate $\mathcal{E}(T(u))$. By (6.1.7) and (6.1.8), it is clear that

$$\mathcal{E}(T(u)) = u\,\mathcal{E}(T(1))$$

and

$$\mathcal{E}(D(t)) = t\,\mathcal{E}(D(1)).$$

Now, by Lemma 1 of Feller (1966, p. 148),

$$\mathcal{E}(T(u)) = \int_0^\infty [1 - \Pr\{T(u) \le t\}]\,dt$$

$$= \int_0^\infty \Pr\{D(t) \le u\}\,dt$$

$$= \int_0^\infty \int_0^u g(y,t)\,dy\,dt,$$

so that

$$\mathcal{E}(T(1)) = \int_0^\infty \int_0^1 g(y,t)\,dy\,dt. \qquad (6.1.13)$$

It is strongly suspected that

$$\mathcal{E}(T(1)) = 1/\mathcal{E}(D(1)), \qquad (6.1.14)$$

although all attempts to prove (or disprove) this fact have failed.

It is obvious that many questions remain unanswered about the process T(u). Possible areas of future research include finding the specific form of (6.1.7), which would completely solve the problem, or at least establishing (6.1.14).

6.2 Definition of the Continuous Inventory System and Its Relation to Previous Results

The inventory problem for the case when we have continuous demand is analogous to the description for the discrete case given in Section 5.1. The biggest difference is that ν, the size of an order of replacement fluid, is not constrained to be an integer. We now discuss the system in detail.

Consider a subwarehouse that maintains a reservoir (inventory) of fluid material. The capacity of the reservoir is S, a positive real number. The fluid in the reservoir is continuously being depleted by its use in some process.

Let $\{D(t);\ t \geq 0\}$ be the stochastic process which represents the demand for the fluid in storage. In keeping with the constraints placed on D(t) in Section 6.1, we assume D(t) is non-negative, continuous, and has an infinitely divisible distribution. The distribution function of D(t) will be given by (6.1.1).

In order to maintain a positive level of fluid on hand, the subwarehouse places an order for an amount ν $(0 < \nu \leq S)$ of replacement fluid to a warehouse. It will be held that the orders are placed at the times $\sigma_1, \sigma_2, \sigma_3, \ldots$, where

$$\sigma_j = \inf \{t \mid D(t) \geq j\nu\}, \qquad j \geq 1. \tag{6.2.1}$$

By Theorem 6.1.1, the random variables $\sigma_2 - \sigma_1, \sigma_3 - \sigma_2, \sigma_4 - \sigma_3, \ldots$, are mutually independent and $\sigma_j - \sigma_{j-1}$ has the distribution function

$$\Pr\{\sigma_j - \sigma_{j-1} \le t\} = \Pr\{D(t) \ge \nu\}$$

$$= \int_{\nu}^{\infty} g(x,t)\, dx, \quad t \ge 0, \quad j \ge 2. \tag{6.2.2}$$

Let $\{S(t);\ t \ge 0\}$ be the stochastic process such that $S(t)$ represents the level of fluid in the reservoir at time t. We then define

$$S(t) = S - D(t) + \nu R(t) \tag{6.2.3}$$

where $R(t)$ is the number of orders that have been filled by the warehouse for our subwarehouse of interest during the interval $[0,t]$.

As before, an order placed at time σ may be one of two types. We have a regular order if $S(\sigma) > S - \nu[S/\nu]$. In this case, the time to fill an order (hereafter, the service time) is assumed to be a random variable. The successive regular service times, denoted by $w_1, w_2, w_3, \ldots$, are assumed to be mutually independent and independent of the demand process $D(t)$. The distribution of w_j is given by

$$\Pr\{w_j \le w\} = H(w), \quad w \ge 0, \quad j \ge 1. \tag{6.2.4}$$

We have an emergency order if $S(\sigma) = S - \nu[S/\nu]$. In this case, $S(\sigma+0) = S - \nu[S/\nu] + \nu$, so that the emergency service time is assumed instantaneous, or at least effectively zero. Utilizing this ordering technique, the reservoir maintained by the subwarehouse is never depleted.

Realistically, the time to process a regular order should depend in some manner on ν, the quantity being ordered. Therefore, one may have $H(w)$ a function of ν.

We assume the subwarehouse places its orders with either a one-server or infinite-server warehouse as described in Section 5.1.

Define

$$F(t,\nu) = \Pr\{D(t) \geq \nu\}$$

$$= \int_{\nu}^{\infty} g(x,t)\, dx, \qquad t \geq 0, \tag{6.2.5}$$

so that $F(\cdot,\nu)$ represents the distribution function of $\sigma_j - \sigma_{j-1}$. As before, we seek that value of ν, the size of the replacement order, such that the cost function $C(\nu)$ defined in Section 5.3 is a minimum. If we replace the distribution functions $G(\cdot)$ and $G_{\nu}(\cdot)$ of Chapter 5 by $F(\cdot,1)$ and $F(\cdot,\nu)$, respectively, then all results of Sections 5.2 through 5.6 hold for our continuous inventory problem. One can easily change the terminology of Definition 5.1.1 so that it expresses the ordering scheme for maintaining the level of fluid for an inventory problem of the continuous type. The main point to remember is that we are now minimizing $C(\nu)$ over all choices of ν in the interval $(0,S]$ and we are not restricted to the integral values of ν as was the case in the last chapter.

An important and realistically relevant example of $D(t)$ is when

$$\mathcal{E}(e^{-\theta D(t)}) = (1 + \theta\rho)^{-t}$$

$$= \exp\,(-t \ln(1 + \theta\rho)).$$

That is, when $D(t)$ has a gamma distribution with parameters ρ and t. In this case,

$$F(t,\nu) = \int_{\nu/\rho}^{\infty} e^{-x}\, x^{t-1}/\Gamma(t)\, dx. \tag{6.2.6}$$

However, no expression for

$$\psi(\theta) = \int_0^\infty e^{-\theta t}\, dF(t,1)$$

could be obtained. Therefore, we are unable to give an example of the calculations involved in utilizing Sections 5.4 through 5.6.

The study of the distribution function given by (6.2.6) is an area of possible future research.

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BIOGRAPHICAL SKETCH

Edwin Luther Bradley, Jr., was born July 16, 1943, at Jacksonville, Florida. In June, 1961, he was graduated from Clay County High School in Green Cove Springs, Florida. In August, 1964, he received the degree of Bachelor of Science with Honors with a major in Mathematics from the University of Florida. He immediately enrolled in the Graduate School of the University of Florida, leaving in January, 1965, to work at Lockheed-Georgia Company at Marietta, Georgia. He returned to Graduate School at the University of Florida in September, 1965, where he worked as a graduate assistant at the University Computing Center as a programmer until November, 1966. He has worked as a graduate assistant in the Department of Statistics from November, 1966, to the present time. During this period he received the degree of Master of Statistics in April, 1967, and from that time he has pursued his work towards the degree of Doctor of Philosophy.

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This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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